1) [25 points] Compute the remainder of

$$2020 \cdot 2021^{2022} + 2023$$

when divided by 7. [Show work, including computations!]

Solution. We have

$$2020 = 288 \cdot 7 + 4.$$

Then:

$$\begin{aligned} 2020 &\equiv 4 \pmod{7}, \\ 2021 = 2020 + 1 &\equiv 4 + 1 = 5 \pmod{7}, \\ 2023 = 2021 + 2 &\equiv 5 + 2 = 0 \pmod{7}. \end{aligned}$$

Hence:

$$2020 \cdot 2021^{2022} + 2023 \equiv 4 \cdot 5^{2022} + 0 \pmod{7}.$$

Now, since 7 is prime, by *Fermat's Little Theorem*, we have that $5^{7-1} = 5^6 \equiv 1 \pmod{7}$, and since

$$2022 = 337 \cdot 6 + 0$$

we have that $2022 \equiv 0 \pmod{6}$. So,

$$5^{2022} \equiv 5^0 = 1 \pmod{7},$$

and

$$2020 \cdot 2021^{2022} + 2023 \equiv 4 \cdot 5^{2022} + 0 \equiv 4 \cdot 1 + 0 = 4 \pmod{7}.$$

So, the remainder is 4. □

2) [25 points] Find all integers x satisfying the congruence/system below. If there is no such integer, explain how you could tell. *You need to show work!* Guessing solutions doesn't yield *any* credit.

(a) Congruence:

$$6x \equiv 11 \pmod{36}.$$

Solution. We have that $\gcd(6, 36) = 6$ and $6 \nmid 11$, so there is no solution. □

(b) System:

$$4x \equiv 7 \pmod{15}$$
$$3x \equiv 12 \pmod{21}.$$

Solution. We have that

$$1 = 4 \cdot 4 + 15 \cdot (-1).$$

So, multiplying the first equation by 4 we get:

$$x \equiv 28 \equiv 13 \pmod{15} \implies x = 13 + 15k, \text{ for some } k \in \mathbb{Z}.$$

Substituting in the second, we get:

$$\begin{aligned} 3 \cdot (13 + 15k) \equiv 12 \pmod{21} &\implies 45k \equiv -27 \pmod{21} \\ &\implies 3k \equiv 15 \pmod{21} \\ &\implies k \equiv 5 \pmod{7} \\ &\implies k = 5 + 7l, \text{ for some } l \in \mathbb{Z}. \end{aligned}$$

Hence,

$$x = 13 + 15k = 13 + 15 \cdot (5 + 7l) = 88 + 105l, \text{ for some } l \in \mathbb{Z}.$$

□

3) [25 points] Prove that there are no integers x, y, z such that $x^4 + y^4 + z^4 = 1234$.

[**Hint:** Fermat's Little Theorem.]

Proof. For $p = 5$ we have that

$$a^4 \equiv \begin{cases} 0, & \text{if } 5 \mid a, \\ 1, & \text{if } 5 \nmid a. \end{cases}$$

So, $x^4 + y^4 + z^4$ can only be 0, 1, 2, or 3 modulo 5. But $1234 \equiv 4 \pmod 5$, so

$$x^4 + y^4 + z^4 \not\equiv 1234 \pmod 5,$$

and hence there are no integers such that $x^4 + y^4 + z^4 = 1234$. □

4) [25 points] Prove that if a and b are relatively prime, then $\gcd(a^2, b^2) = 1$.

[**Note:** This was a HW problem.]

Proof. If $\gcd(a^2, b^2) \neq 1$, then there is a *prime* p that is a common divisor. (This was another HW problem, but we can just take a prime divisor of the GCD.) Since $p \mid a^2$, we have that $p \mid a$ by Euclid's Lemma. Similarly, since $p \mid b^2$, we have that $p \mid b$. But this means that p is a common divisor of a and b, which is a contradiction since $p \geq 2$. □