1) True of False? [If true, give a proof. If false, give a counter-example.] Remember that $U(R) = R^{\times}$ = set of units of R.
[**Note:** This was a HW problem.]

(a) [10 points] If R is an infinite commutative ring, then $U(R)$ is also infinite.

Solution. False. $\mathbb{Z}$ is infinite but $U(\mathbb{Z}) = \{1, -1\}$ is finite. □

(b) [15 points] If S is a subring of the commutative ring R, then $U(S) = U(R) \cap S$.

Solution. False. Let $R = \mathbb{Q}$ and $S = \mathbb{Z}$. Then

$$U(R) \cap S = (\mathbb{Q} \setminus \{0\}) \cap \mathbb{Z} = \mathbb{Z} \setminus \{0\} \neq \{1, -1\} = U(S).$$

□

2) Let R be a domain and $F = \mathrm{Frac}(R)$.
[**Note:** Both were done in class!]

(a) [10 points] Prove that for all $a, b, c \in R$, with $b, c \neq 0$, we have that $\dfrac{a}{b} = \dfrac{ac}{bc}$.

Proof. We have that $a \cdot (bc) = (ab) \cdot c = (ba) \cdot c = b \cdot (ac)$ since R is commutative. □

(b) [15 points] Prove that for all $a, b, c \in R$, with $c \neq 0$, we have that $\dfrac{a}{c} + \dfrac{b}{c} = \dfrac{a+b}{c}$.

Proof. We have, by definition that:

$$\frac{a}{c} + \frac{b}{c} = \frac{ac + bc}{c^2} = \frac{(a+b) \cdot c}{c \cdot c} = \frac{a+b}{c},$$

with the last equality true by the previous part.

Alternatively: We have $\dfrac{ac+bc}{c^2} = \dfrac{a+b}{c}$ since $(ac+bc) \cdot c = ac^2 + bc^2 = (a+b) \cdot c^2$. □

3) Answer the following questions. [*No justification needed.* But you *can* give justification, which can help with partial credit.]

(a) [5 points] The units of $\mathbb{Z}/8\mathbb{Z}$ are: $\boxed{1, 3, 5, 7}$. [Since $8 = 2^3$, we have that $\gcd(a, 8) = 1$ iff a is odd, and the units are the elements $a \in \mathbb{Z}/8\mathbb{Z}$ with $\gcd(a, 8) = 1$.]

(b) [5 points] The prime field of $\mathbb{C}$ is: $\boxed{\mathbb{Q}}$. [We have that $\mathrm{char}(\mathbb{C}) = 0$, so its prime field is $\mathbb{Q}$.]

(c) [5 points] The prime field of $\mathbb{F}_{11}$ is: $\boxed{\mathbb{F}_{11}}$. [We have that $\mathrm{char}(\mathbb{F}_{11}) = 11$, so its prime field is $\mathbb{F}_{11}$.]

(d) [5 points] The field of fractions of $\mathbb{R}$ is: $\boxed{\mathbb{R}}$. [Since $\mathbb{R}$ is itself a field, it is its own field of fractions.]

(e) [5 points] The characteristic of $\mathbb{Z}/12\mathbb{Z}$ is: $\boxed{12}$. [The smallest positive integer n such $n \cdot 1 = 0$ in $\mathbb{Z}/12\mathbb{Z}$ is 12.]

4) [25 points] Let R be a commutative ring. Prove that if $a \in R$, then $(-1) \cdot a = -a$ *using only the commutative ring axioms* [provided in the last page] *and* the fact that $0 \cdot a = 0$ for all $a \in R$.
[**Hints:** This was a done in class! Note that it suffices to show that $a + ((-1) \cdot a) = 0$.]

Solution. We have

$$\begin{aligned} a + ((-1) \cdot a) &= 1 \cdot a + ((-1) \cdot a) && \text{[by Axiom 7]} \\ &= (1 + (-1)) \cdot a && \text{[by Axiom 8]} \\ &= 0 \cdot a && \text{[by Axiom 4]} \\ &= 0 && \text{[Previous Result]} \end{aligned}$$

Hence, $(-1) \cdot a = -a$.

□

Commutative Ring Axioms: A [non-empty] set with two operations, $+$ and $\cdot$, is a commutative ring if:

0. For all $a, b \in R$ we have that $a + b \in R$ and $a \cdot b \in R$.
1. For all $a, b \in R$ we have that $a + b = b + a$.
2. For all $a, b, c \in R$ we have that $(a + b) + c = a + (b + c)$.
3. There exists $0 \in R$ such that for all $a \in R$ we have $a + 0 = a$.
4. For all $a \in R$ there exists $-a \in R$ such that $a + (-a) = 0$.
5. For all $a, b \in R$ we have that $a \cdot b = b \cdot a$.
6. For all $a, b, c \in R$ we have that $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
7. There is $1 \in R$ such that for all $a \in R$ we have that $1 \cdot a = a$
8. For all $a, b, c \in R$ we have that $a \cdot (b + c) = a \cdot b + a \cdot c$