FINAL

M559 – LINEAR ALGEBRA – MAY 10TH, 2024

Solve all problems in class. [I hope you can do all of this in class, as it is comparable to a diagnostic exam.] Then, you may take this sheet home and solve all remaining problems, or problems that you think you've missed, upload it to Canvas by Sunday (05/12) by 11:59pm. I will consider it for some partial credit. [At *most* half of the original number of points you've missed in the question.]

Important: After you take them home, you should treat these problems as a *take-home exam*, not as a homework. So, you should not discuss *anything* about these problems with *anyone* (except me), nor use any texts or the internet.

1. Let T be a linear operator on a vector space V of [finite] dimension n. Suppose that $\operatorname{rank}(T^2) = \operatorname{rank}(T)$. Prove that $\ker(T) \cap \operatorname{im}(T) = \{\vec{0}\}$.

2. Given an example of two real 4×4 *nilpotent* matrices that have the same minimal and characteristic polynomials, but are not similar. **Justify!**

3. Let A be a 3×3 matrix over $\mathbb{R}$ with eigenvalues 0, 1, and 2, and $B \overset{\text{def}}{=} A^2 + I$. Prove that B is invertible.

4. Let A be a 2024×2024 matrix over $\mathbb{C}$ such that $A^3 = A$ and $\operatorname{tr}(A) \geq 2024$, where $\operatorname{tr}(A)$ is the *trace* of A [the sum of all entries on the main diagonal of A]. Prove that A is the identity matrix.

[**Hint:** We showed in the homework that if A and B are similar, then they have the same trace. You can use this result without proving it.]

5. Let A be an $n \times n$ matrix with characteristic polynomial

$$\chi_A = (x - c_1)^{d_1}(x - c_2)^{d_2} \cdots (x - c_k)^{d_k},$$

with c_i's distinct and $d_i \geq 1$. Find $\det(A)$ and $\operatorname{tr}(A)$. **Justify your answer!**

6. Let A be a Hermetian [i.e., self-adjoint] $n \times n$ matrix with complex entries. Prove that for all $v \in \mathbb{C}^n$ [seen as a *column* vector], we have that $v^*Av \in \mathbb{R}$. [As usual, B^* denotes the *adjoint* of B, i.e., the transpose of the complex conjugate of B.]