# FINAL

M559 – LINEAR ALGEBRA – MAY 10TH, 2024

**1.** Let $T$ be a linear operator on a vector space $V$ of [finite] dimension $n$. Suppose that $\mathrm{rank}(T^2) = \mathrm{rank}(T)$. Prove that $\ker(T) \cap \mathrm{im}(T) = \{\vec{0}\}$.

*Proof.* First, observe that clearly $\mathrm{im}\, T^2 \subseteq \mathrm{im}\, T$. Since $\mathrm{rank}(T^2) = \mathrm{rank}(T)$, we have that $\mathrm{im}\, T^2 = \mathrm{im}\, T$.

Then, we have that

$$\begin{aligned} n &= \dim \ker(T) + \mathrm{rank}(T) && \text{[Rank-Nullity Theorem for } T\text{]} \\ &= \dim \ker(T) + \mathrm{rank}(T^2) && \text{[since } \mathrm{rank}(T) = \mathrm{rank}(T^2)\text{]} \\ &= \dim \ker(T^2) + \mathrm{rank}(T^2) && \text{[Rank-Nullity Theorem for } T^2\text{]} \end{aligned}$$

and hence $\dim \ker(T^2) = \dim \ker(T)$. Now, we also clearly have $\ker(T) \subseteq \ker(T^2)$, we have that $\ker(T) = \ker(T^2)$. So, if $v \in \ker(T) \cap \mathrm{im}(T)$, then $v = T(W)$ for some $w \in V$, and $\vec{0} = T(v) = T^2(w)$. So, $w \in \ker(T^2) = \ker(T)$ and $v = T(w) = \vec{0}$. □

**2.** Given an example of two real $4 \times 4$ *nilpotent* matrices that have the same minimal and characteristic polynomials, but are not similar. **Justify!**

*Solution.* We have that

$$A \stackrel{\text{def}}{=} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad B \stackrel{\text{def}}{=} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

are both in Jordan form, from which we get that $\chi_A = \chi_B = x^4$, so both are nilpotent, and from the first $2 \times 2$ elementary Jordan block, we get that $\mu_A = \mu_B = x^2$, but since they have different Jordan forms they are not similar [or note that the first has rank 2 and the second has rank 1]. □

**3.** Let $A$ be a $3 \times 3$ matrix over $\mathbb{R}$ with eigenvalues 0, 1, and 2, and $B \stackrel{\text{def}}{=} A^2 + I$. Prove that $B$ is invertible.

*Proof.* Let $\mathcal{B} = \{v_1, v_2, v_3\}$, where $v_1$, $v_2$, $v_3$ are eigenvectors associated to 0, 1, and 2, respectively. Since $A$ is $3 \times 3$, $\mathcal{B}$ is a basis of $\mathbb{R}^3$. Then, we have that

$$
\begin{aligned}
B(v_1) &= A^2(v_1) + v_1 = v_1, \\
B(v_2) &= A^2(v_2) + v_2 = 2v_2, \\
B(v_3) &= A^2(v_3) + v_3 = 5v_3,
\end{aligned}
$$

and hence $\{1, 2, 5\} \subseteq \operatorname{spec}(B)$, and since $B$ is $3 \times 3$, we have equality. So, $0 \notin \operatorname{spec}(B)$, so $\ker(B) = \{\vec{0}\}$ and $B$ is invertible. □

**4.** Let $A$ be a $2024 \times 2024$ matrix over $\mathbb{C}$ such that $A^3 = A$ and $\operatorname{tr}(A) \geq 2024$, where $\operatorname{tr}(A)$ is the *trace* of $A$ [the sum of all entries on the main diagonal of $A$]. Prove that $A$ is the identity matrix.
[**Hint:** We showed in the homework that if $A$ and $B$ are similar, then they have the same trace. You can use this result without proving it.]

*Proof.* We have that $x^3 - x = x(x-1)(x+1)$ annihilates the matrix, so the minimal polynomial $\mu_A$ divides $x(x-1)(x+1)$, so it has simple roots, and hence $A$ is diagonalizable, with either 0, 1, or $-1$ in its diagonal.

If $P$ is an invertible matrix such that $D = P^{-1}AP$ is the diagonal form, then $\operatorname{tr}(A) = \operatorname{tr}(D)$ is a sum of $n$ numbers belonging to $\{0, 1, -1\}$, so $\operatorname{tr}(A) \leq n$. Since $\operatorname{tr}(A) \geq n$, we must have the equality and that all diagonal entries of $D$ are 1, i.e., $D = I$, which implies that $A = PDP^1 = PIP^{-1} = I$. □

**5.** Let $A$ be an $n \times n$ matrix with characteristic polynomial

$$\chi_A = (x - c_1)^{d_1}(x - c_2)^{d_2} \cdots (x - c_k)^{d_k},$$

with $c_i$'s distinct and $d_i \geq 1$. Find $\det(A)$ and $\operatorname{tr}(A)$. **Justify your answer!**

*Solution.* Since the characteristic polynomial factors as product of linear polynomials, we have a Jordan form. The Jordan block of $(x - c_i)^{d_i}$ is upper triangular and has $d_i$ $c_i$'s in its diagonal. Hence:

$$
\begin{aligned}
\det(A) &= c_1^{d_1} \cdots c_k^{d_k}, \\
\operatorname{tr}(A) &= d_1 c_1 + \cdots + d_k c_k.
\end{aligned}
$$

□

**6.** Let $A$ be a Hermetian [i.e., self-adjoint] $n \times n$ matrix with complex entries. Prove that for all $v \in \mathbb{C}^n$ [seen as a *column* vector], we have that $v^* A v \in \mathbb{R}$. [As usual, $B^*$ denotes the adjoint of $B$, i.e., the transpose of the complex conjugate of $B$.]

*Proof.* Since $A^* = A$, we have

$$v^* A v = \langle Av, v \rangle = \langle v, A^* v \rangle = \langle v, Av \rangle = \overline{\langle Av, v \rangle}.$$

Hence, since $v^* A v$ is equal to both $\langle Av, v \rangle$ and $\overline{\langle Av, v \rangle}$, it must be real. □