MIDTERM SOLUTION

M559 – LINEAR ALGEBRA

1. Let V be a vector space over the field F. Prove that if $S = \{v_1, v_2, \dots, v_n\} \subseteq V$ is such that $V = \operatorname{span}(S)$, but for all $i \in \{1, 2, \dots, n\}$ we have that $V \neq \operatorname{span}(S \setminus \{v_i\})$, then S is a basis of V.

Proof. Short proof:

It suffices to show that S is linearly independent. So, assume it is not. Then, since $\operatorname{span}(S) = V$ and S is linearly dependent, we have can remove some vector $v_i \in S$ and still have that $\operatorname{span}(S \setminus \{v_i\}) = \operatorname{span}(S) = V$. [This was proved in class: we can always remove an element of a linearly dependent set without changing the space that they generate.] But this is a contradiction, so S must be linearly independent.

Alternative proof: We can basically replicate the proof of the statement mentioned above.

It suffices to show that S is linearly independent. So, assume that there $c_1, \dots, c_n \in F$ such that

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = \vec{0}.$$

If $c_i \neq 0$, then

$$v_i = -\frac{c_1}{c_i} v_1 - \frac{c_2}{c_i} v_2 - \cdots - \frac{c_{i-1}}{c_i} v_{i-1} - \frac{c_{i+1}}{c_i} v_{i+1} - \cdots - \frac{c_n}{c_i} v_n,$$

and hence $v_i \in \operatorname{span}(S \setminus \{v_i\})$, and hence $\operatorname{span}(S \setminus \{v_i\}) = \operatorname{span}(S) = V$, a contradiction. Hence we must have that $c_i = 0$. Since i was arbitrary, we have $c_1 = c_2 = \cdots = c_n = 0$.

□

2. Let V and W be vector spaces over the field F of [finite] dimensions n and m respectively, $T : V \to W$ and $S : W \to V$ be linear transformations such that $T \circ S$ and $S \circ T$ are the identity maps of W and V respectively.

(a) Show that both T and S are onto.

Proof. Short proof: We have that T and S are inverses of each other, so they are bijections, and in particular, they are onto.

Alternative proof: We can prove the onto part of the above result directly:
Let $w \in W$. Then $w = (T \circ S)(w) = T(S(w))$, so $w \in \mathrm{im}(T)$.
Similarly, if $v \in V$. Then $v = (S \circ T)(v) = S(T(v))$, so $v \in \mathrm{im}(S)$. $\square$

(b) Show that $m = n$.

Proof. Short proof: If you proved that T and S are bijections, then they are isomorphisms, so V and W are isomorphic and isomorphic spaces have the same dimension.

Alternative proof: Since T is onto we have that $\mathrm{rank}(T) = \dim W = m$. So, $0 \leq \dim\ker(T) = n - \mathrm{rank}(T) = n - m$, and hence $n \geq m$.
Similarly, since S is onto we have that $\mathrm{rank}(S) = \dim V = n$. So, $0 \leq \dim\ker(S) = m - \mathrm{rank}(S) = m - n$, and hence $m \geq n$. These two inequalities give that $m = n$. $\square$

3. Let V be a vector space over F [possibly infinite dimensional] and $f \in V^* \setminus \{0\}$. Let v_0 such that $f(v_0) \neq 0$ and $N \stackrel{\text{def}}{=} \ker(f)$. Prove that for all $v \in V$ there are *unique* $c \in F$ and $w \in N$ such that $v = cv_0 + w$.
[**Hint:** Take $v \in V$ and find some $c \in F$ such that $f(v - cv_0) = 0$.]

Proof. Short proof: We've seen in class that N [as above] is a hyperspace, so for all $w \notin N$, we have that $V = \mathrm{span}(\{w\}) + N$. In particular, since $v_0 \notin N$, we have that $V = \mathrm{span}(\{v_0\}) + W$, so for all $v \in V$, there are $c \in F$ and $w \in N$ such that $v = cv_0 + w$.

Now suppose that $c, c' \in F$ and $w, w' \in N$ are such that $cv_0 + w = c'v_0 + w'$. Then, $(c - c')v_0 = w' - w \in N$. Hence,

$$0 = f(w' - w) = f((c - c')v_0) = (c - c_0)f(v_0),$$

and since $f(v_0) \neq 0$, we must have that $c = c'$. But then $\vec{0} = w' - w$, i..e. $w = w'$.

Alternative proof: Since $f(v_0) \neq 0$, we have that

$$f\left(v - \frac{f(v)}{f(v_0)v_0}\right) = f(v) - \frac{f(v)}{f(v_0)}f(v_0) = f(v) - f(v) = 0.$$

So, if $c \stackrel{\text{def}}{=} \dfrac{f(v)}{f(v_0)}$, then $w \stackrel{\text{def}}{=} v - cv_0 \in N$. Then, $v = cv_0 + w$.
The proof uniqueness is the same as above.

$\square$