1) [20 points] *Quickies!* [These should take you 10 seconds each if you've studied.] You don't need to justify your answers.

(a) If a square matrix is not invertible, what can we say about its reduced row echelon form? [Simple and short answer!]

Answer: It has a row of zeros. [Or, it has a column with no leading one, or it is not the identity. All are valid.]

(b) $\begin{vmatrix} 2 & \sqrt{3} & -\pi & 12/131 \\ 0 & -1 & e^2 & \ln(10) \\ 0 & 0 & 7 & \cos(3) \\ 0 & 0 & 0 & 1 \end{vmatrix} =$

Answer: $= 2 \cdot (-1) \cdot 7 \cdot 1 = -14$. [Determinant of upper triangular matrix.]

(c) $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{bmatrix} =$

Answer: $= \begin{bmatrix} 2 & -1 & 5 \\ 4 & -2 & 10 \\ 6 & -3 & 15 \end{bmatrix}$ [Multiplication by diagonal matrix.]

(d) If $E \cdot \begin{bmatrix} 2 & -1 & 3 \\ 0 & 2 & 0 \\ 5 & -4 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 3 \\ -5 & 6 & -1 \\ 5 & -4 & 1 \end{bmatrix}$, then $E =$

Answer: $= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$. [Multiplication by elementary matrix.]

(e) If $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 3$, then $\begin{vmatrix} g & h & i \\ d-g & e-h & f-i \\ 5a & 5b & 5c \end{vmatrix} =$

Answer: $3 \cdot (-1) \cdot 5 \cdot 1 = -15$ [(switch 1st and 3rd rows), (multiply 3rd row by 5), (add -1 times the 1st row to the 2nd)].

2) [15 points] Let

$$A = \begin{bmatrix} 1 & 0 \\ 1 & -1 \\ 3 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & -1 \\ 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

Compute $\operatorname{tr}((A^{\mathrm{T}} \cdot B)^{-1})$.

Solution. We have:

$$A^{\mathrm{T}} = \begin{bmatrix} 1 & 1 & 3 \\ 0 & -1 & 1 \end{bmatrix}, \qquad A^{\mathrm{T}} \cdot B = \begin{bmatrix} 3 & 5 \\ -1 & 2 \end{bmatrix}, \quad (A^{\mathrm{T}} \cdot B)^{-1} = \frac{1}{11} \begin{bmatrix} 2 & -5 \\ 1 & 3 \end{bmatrix}$$

So, $\operatorname{tr}((A^{\mathrm{T}} \cdot B)^{-1}) = 2/11 + 3/11 = 5/11$.

□

3) [15 points] Let

$$A = \begin{bmatrix} 2 & 0 & 1 \\ -2 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

Find C_{21}, the cofactor of A at position $(2,1)$ [or at a_{21}, as the book writes] and $(\mathrm{adj}(A))_{12}$, i.e., the entry at position $(1,2)$ of the adjoint of A.

Solution. We have:

$$C_{21} = (-1)^{2+1} \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = (-1) \cdot (-1) = 1.$$

Since $\mathrm{adj}(A) = (C_{ij})^{\mathrm{T}}$ [i.e., $(\mathrm{adj}(A))_{ij} = C_{ji}$], we have that the entry at position $(1,2)$ of the adjoint is $C_{21} = 1$.

□

4) [25 points] Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}.$$

Find $\det(2 \cdot A^6)$ and A^{-1}. [Don't work harder than you have to!]

Solution. Finding the inverse of A is Example 4 on pg. 55. [Just look at the solution there.] We have:

$$A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}.$$

To find the determinant of A [not of $2 \cdot A^6$ yet] we may simply see what were the row operations used: looking at the text, we see that the only operation that changes the determinant is one multiplication by -1. So the determinant of A is -1.

Hence, $\det(2 \cdot A^6) = 2^3 \det(A^6) = 8 \cdot (\det(A))^6 = 8 \cdot (-1)^6 = 8$.

□

5) [25 points] Let

$$A = \begin{bmatrix} 2 & 4 & 1 & 0 \\ 1 & 2 & 0 & 0 \\ -1 & -2 & -1 & 1 \\ 4 & 8 & 2 & -1 \end{bmatrix}, \quad \mathbf{b}_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 0 \\ -1 \\ -3 \\ -1 \end{bmatrix}$$

Solve the systems $A\mathbf{x} = \mathbf{b}_1$ and $A\mathbf{x} = \mathbf{b}_2$.

Solution. We solve the systems together:

$$\left[\begin{array}{rrrr|r|r} 2 & 4 & 1 & 0 & 2 & 0 \\ 1 & 2 & 0 & 0 & 1 & -1 \\ -1 & -2 & -1 & 1 & 2 & -3 \\ 4 & 8 & 2 & -1 & 1 & -1 \end{array}\right] \sim \left[\begin{array}{rrrr|r|r} 1 & 2 & 0 & 0 & 1 & -1 \\ 2 & 4 & 1 & 0 & 2 & 0 \\ -1 & -2 & -1 & 1 & 2 & -3 \\ 4 & 8 & 2 & -1 & 1 & -1 \end{array}\right]$$

$$\sim \left[\begin{array}{rrrr|r|r} 1 & 2 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & -1 & 1 & 3 & -4 \\ 0 & 0 & 2 & -1 & -3 & 3 \end{array}\right]$$

$$\sim \left[\begin{array}{rrrr|r|r} 1 & 2 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & -1 & -2 & -1 \end{array}\right] \sim \left[\begin{array}{rrrr|r|r} 1 & 2 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 & 0 & -3 \end{array}\right]$$

Hence, the second system has no solution [last row gives $0 = -3$], and the first system has solution:

$$x_1 = 1 - 2t, \quad x_2 = t, \quad x_3 = 0, \quad x_4 = 3,$$

where t can be any real number [free parameter]. □