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Math 251

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Name: ....

Student ID (last 6 digits): XXX- ....

Midterm 2

You have 60 minutes to complete the exam. Do all work on this exam, i.e., on the page of the respective assignment. Indicate clearly, when you continue your solution on the back of the page or another part of the exam.

Write your name and the last six digits of your student ID number on the top of this page. Check that no pages of your exam are missing. This exam has 6 questions and 9 printed pages (including this one, a page with the vector space axioms, and a page for scratch work in the end).

No books, notes or calculators are allowed on this exam!

Show all work! (Unless I say otherwise.) Correct answers without work will receive **zero**. Also, **points will be taken from messy solutions**.

Good luck!

Question	Max. Points	Score
1	20	
2	15	
3	15	
4	15	
5	15	
6	20	
Total	100	

1) [20 points] *Quickies!* [These should take you 10 seconds each if you've studied.] You don't need to justify your answers.

(a) If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a *one-to-one* linear transformation, then what can we say about the matrix $[T]$?

(b) If $T : \mathbb{R}^3 \to \mathbb{R}^2$ is such that $T(\mathbf{e}_1) = (2, 1)$, $T(\mathbf{e}_2) = (1, 0)$, and $T(\mathbf{e}_3) = (-1, 3)$, what is $[T]$?

(c) If A is an m by n matrix, then what are the domain and codomain of the linear transformation T_A?

(d) If $T : \mathbb{R}^2 \to \mathbb{R}$ is linear with $T(\mathbf{u}) = 2$ and $T(\mathbf{v}) = 3$, then what is $T(\mathbf{u} + 2\mathbf{v})$?

(e) If the projection of $\mathbf{v}$ on the direction of $\mathbf{u}$ is $(2, 0, 3)$ and the component of $\mathbf{v}$ orthogonal to $\mathbf{u}$ is $(-1, 1, -2)$, then what is $\mathbf{v}$? [This is easier than it might seem at first! *There is very little computation involved!*]

2) [15 points] Let $\mathbf{u} = (1, 1, 3)$ and $\mathbf{v} = (0, -2, 0)$.

(a) Find the cosine of the angle between $\mathbf{u}$ and $\mathbf{v}$.

(b) Compute $\operatorname{proj}_{\mathbf{u}} \mathbf{v}$.

3) [15 points] Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the reflection about the line $y = 2x$.

(a) Find all eigenvalues and the eigenvectors associated to each eigenvalue.

(b) Is the matrix $[T]$ invertible? [Justify!]

[**Hint:** You do *not* need to compute $[T]$ in this question!]

4) [15 points] Let $S = \{(2,3,6),(4,1,7),(6,2,11)\}$. This set is not linearly independent and does not span $\mathbb{R}^3$. [Just take my word for it!] Find a linear combination of the vectors of S that give the zero vector *with at least one coefficient being non-zero* and a vector not in $\operatorname{span}(S)$.
[**Hint:** You've done both in HW. Be careful to not do the same computation twice!]

5) [15 points] Is the set of all 2 by 2 matrices of the form

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, \qquad \text{with } a, b \in \mathbb{R},$$

a vector space with the usual addition and scalar multiplication of matrices? [Justify!]

6) [20 points] Let $V = (0, +\infty)$ [i.e., all positive real numbers] with the following operations:

$$x \oplus y = xy, \qquad k \odot x = x^k \qquad [x, y \in V,\ k \in \mathbb{R}]$$

[Just like in your HW! I am using "$\oplus$" and "$\odot$" to denote the sum and scalar multiplication to avoid confusion with the regular operations, though.] The set V with these operations *is* a vector space. [Take my word for it.]

(a) Check that $k \odot (x \oplus y) = (k \odot x) \oplus (k \odot y)$.

(b) What is the "zero" of this vector space [i.e., the element $\theta \in V$ such that $\theta \oplus x = x$ for all $x \in V$]?

(c) Given $x \in V$, what is its "negative" of x [i.e., the element $y \in V$ such that $x \oplus y = \theta$, where θ is the zero from the previous item]?

Vector Space Axioms

A non-empty set V with a sum and a scalar product is a vector space if it satisfies the following conditions:

0. $\mathbf{u} + \mathbf{v} \in V$ for all $\mathbf{u}, \mathbf{v} \in V$, and $k\mathbf{u} \in V$ for all $\mathbf{u} \in V$ and $k \in \mathbb{R}$;
1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ for all $\mathbf{u}, \mathbf{v} \in V$;
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$;
3. there is $\mathbf{0} \in V$ such that $\mathbf{0} + \mathbf{u} = \mathbf{u}$ for all $\mathbf{u} \in V$;
4. given $\mathbf{u} \in V$, there exists $-\mathbf{u} \in V$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$;
5. $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$ for all $\mathbf{u}, \mathbf{v} \in V$ and $k \in \mathbb{R}$;
6. $(k + l)\mathbf{u} = k\mathbf{u} + l\mathbf{u}$ for all $\mathbf{u} \in V$ and $k, l \in \mathbb{R}$;
7. $k(l\mathbf{u}) = (kl)\mathbf{u}$ for all $\mathbf{u} \in V$ and $k, l \in \mathbb{R}$;
8. $1\mathbf{u} = \mathbf{u}$ for all $\mathbf{u} \in V$.

Scratch: