July 2nd, 2009

Math 251

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Summer 2009

Name: ....

Student ID (last 6 digits): XXX- ....

Final

You have one hour and a half to complete the exam. Do all work on this exam, i.e., on the page of the respective assignment. Indicate clearly, when you continue your solution on the back of the page or another part of the exam.

Write your name and the last six digits of your student ID number on the top of this page. Check that no pages of your exam are missing. This exam has 5 questions and 10 printed pages (including this one and a page for scratch work in the end).

No books, notes or calculators are allowed on this exam!

Show all work! (Unless I say otherwise.) Even correct answers without work may result in point deductions. Also, **points will be taken from messy solutions**.

Good luck!

Question	Max. Points	Score
1	10	
2	15	
3	40	
4	15	
5	20	
Total	100	

1) [10 points] Put the following matrix in *reduced* row echelon form:

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix}$$

2) [15 points] Let

$$A = \begin{bmatrix} 4 & 1 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & 1 & -1 & -3 \\ 2 & 1 & 2 & 3 & 0 \\ 1 & 2 & 0 & 2 & 6 \end{bmatrix}.$$

Compute $\det(A)$.

3) [40 points] You should be able to answer the following questions *quickly*. You do *not* need to justify your answers.

(a) [4 points] Give the matrix that represents the rotation by $\pi/2$ about the z-axis, followed by a reflection about the xz-plane in $\mathbb{R}^3$.

(b) [3 points] If $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions, how many solutions can $A\mathbf{x} = \mathbf{b}$ possibly have?

(c) [3 points] If A is an invertible n by n matrix, then what can we say about the reduced echelon form of A.

(d) [3 points] Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be the linear transformation given by

$$T(x_1, x_2, x_3, x_4) = (2x_1, -x_2, x_3, 3x_4).$$

Give $[T^{-1}]$.

(e) [3 points] Let T_A be the a linear transformation associated to the m by n matrix A. If T_A is onto, then what can we say about the rank of A? [If this rank is unrelated to whether or not T_A is onto, just say so.]

(f) [3 points] Is $\{1+x^2, 2-x^3, 1+x+x^2+x^3\}$ a basis of P_3? Justify your answer in one short sentence.

(g) [3 points] If $B = \{\mathbf{e}_1, \mathbf{e}_2\}$ [the standard basis of $\mathbb{R}^2$] and $B' = \{(1,1),(2,1)\}$, find the transition matrix from B to B'.

(h) [3 points] If $S = \{(1,0,1),(-2,1,1),(0,0,3)\}$ is a basis of $\mathbb{R}^3$, then the coordinates $((2,2,2))_S$ is given by the solution of what linear system? Give your answer in matrix form.

(i) [3 points] What is the dimension of $M_{m\times n}$?

(j) [3 points] Give the standard basis of P_3.

(k) [3 points] Let $S = \{\mathbf{v}_1, \mathbf{v}_2\}$ be an orthogonal, *but not orthonormal*, basis of a subspace W of V, and $\mathbf{v} \in V$, give the formula for $\operatorname{proj}_W \mathbf{v}$.

(l) [3 points] If A is a 5 by 4 matrix of rank 3, give the nullities of A and A^{T}.

(m) [3 points] What condition on the size of the matrix A guarantee that the system $A\mathbf{x} = \mathbf{0}$ has a non-trivial solution? [If there is no such condition, just say so.]

4) [15 points] Let

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

(a) [3 points] Find the eigenvalues of A. [You do *not* need to justify this one.]

(b) [6 points] Find the eigenspaces associated to each eigenvalue.

(c) [6 points] Is A diagonalizable? If so, give P such that $P^{-1}AP$ is diagonal *and* the resulting diagonal form. [You do *not* need to justify in this case.] If not, explain why not.

5) [20 points] Let

$$\begin{aligned}
\mathbf{v}_1 &= (4, -4, 2, 2, 4, 1, 17),\\
\mathbf{v}_2 &= (-1, 1, -1, 1, -1, -1, -6),\\
\mathbf{v}_3 &= (3, -3, 2, 0, 3, 1, 14),\\
\mathbf{v}_4 &= (10, -10, 5, 5, 10, 3, 43),\\
\mathbf{v}_5 &= (2, -2, 1, 1, 2, 1, 9),
\end{aligned}$$

and $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$, and $V = \operatorname{span}(S)$. Given that

$$\begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \\ \mathbf{v}_4 \\ \mathbf{v}_5 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 2 & 1 & 0 & 3 \\ 0 & 0 & 1 & -3 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{v}_5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 3 & 1 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

answer the questions below. [You do *not* need to justify any of the items below.]

(a) [4 points] What are the dimension of V and $V^{\perp}$ [the orthogonal complement of V in $\mathbb{R}^7$]?

(b) [4 points] Find a basis for V.

(c) [4 points] Find a basis for $V^{\perp}$.

(d) [4 points] If possible, find a non-trivial linear combination [i.e., not all coefficients equal to zero] of the elements of S which give the zero vector of $\mathbb{R}^7$. [**Hint:** Start by writing a vector of S as a linear combination of the others.]

(e) [4 points] Which vectors from the standard basis of $\mathbb{R}^7$ you can add to the vectors in the basis of V you've found above to obtain a basis of all of $\mathbb{R}^7$?

Scratch: