1) [10 points] Put the following matrix in *reduced* row echelon form:

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix}$$

Solution. This is the coefficient matrix of the system in Example 4 on pg. 12 of the text. [Just follow the steps disregarding the last column.] The reduced echelon form is:

$$\begin{bmatrix} 1 & 3 & 0 & 4 & 2 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

□

2) [15 points] Let

$$A = \begin{bmatrix} 4 & 1 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & 1 & -1 & -3 \\ 2 & 1 & 2 & 3 & 0 \\ 1 & 2 & 0 & 2 & 6 \end{bmatrix}.$$

Compute $\det(A)$.

Solution. We have:

$$\begin{aligned}
\det(A) &= 1 \cdot \begin{vmatrix} 4 & 1 & 0 & 3 \\ 1 & -1 & 1 & -3 \\ 2 & 1 & 2 & 0 \\ 1 & 2 & 0 & 6 \end{vmatrix} && \text{[cofactors through the 2nd row]} \\
&= (-1) \cdot \begin{vmatrix} 4 & 1 & 3 \\ 2 & 1 & 0 \\ 1 & 2 & 6 \end{vmatrix} + 2 \cdot \begin{vmatrix} 4 & 1 & 3 \\ 1 & -1 & -3 \\ 1 & 2 & 6 \end{vmatrix} && \text{[cofactors through the 3rd col]} \\
&= (-1) \cdot (24 + 12 - (3 + 12)) + 2 \cdot 0 && \text{[Sarrus rule and col. mult. of another]} \\
&= -21.
\end{aligned}$$

□

3) [40 points] You should be able to answer the following questions *quickly*. You do *not* need to justify your answers.

(a) [4 points] Give the matrix that represents the rotation by $\pi/2$ about the z-axis, followed by a reflection about the xz-plane in $\mathbb{R}^3$.

Solution. We have:

$$\begin{array}{ccccc} \mathbf{e}_1 & \longrightarrow & \mathbf{e}_2 & \longrightarrow & -\mathbf{e}_2, \\ \mathbf{e}_2 & \longrightarrow & -\mathbf{e}_1 & \longrightarrow & -\mathbf{e}_1, \\ \mathbf{e}_3 & \longrightarrow & \mathbf{e}_3 & \longrightarrow & \mathbf{e}_3. \end{array}$$

So, the matrix is $\begin{bmatrix} -\mathbf{e}_2 & -\mathbf{e}_1 & \mathbf{e}_3 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

□

(b) [3 points] If $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions, how many solutions can $A\mathbf{x} = \mathbf{b}$ possibly have?

Solution. It could have infinitely many or none at all. □

(c) [3 points] If A is an invertible n by n matrix, then what can we say about the reduced echelon form of A.

Solution. It is the identity matrix I_n. □

(d) [3 points] Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be the linear transformation given by

$$T(x_1, x_2, x_3, x_4) = (2x_1, -x_2, x_3, 3x_4).$$

Give $[T^{-1}]$.

Solution.

$$[T^{-1}] = [T]^{-1} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 1/2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/3 \end{bmatrix}.$$

□

(e) [3 points] Let T_A be the a linear transformation associated to the m by n matrix A. If T_A is onto, then what can we say about the rank of A? [If this rank is unrelated to whether or not T_A is onto, just say so.]

Solution. The rank must be m. □

(f) [3 points] Is $\{1+x^2, 2-x^3, 1+x+x^2+x^3\}$ a basis of P_3? Justify your answer in one short sentence.

Solution. No, since $\dim(P_3) = 4$ and we only have 3 vectors in the set. □

(g) [3 points] If $B = \{\mathbf{e}_1, \mathbf{e}_2\}$ and $B' = \{(1,1), (2,1)\}$, find the transition matrix from B to B'.

Solution. The matrix is:

$$\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^{-1} = \frac{1}{-1} \cdot \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}.$$

□

(h) [3 points] If $S = \{(1,0,1), (-2,1,1), (0,0,3)\}$ is a basis of $\mathbb{R}^3$, then the coordinates $((2,2,2))_S$ is given by the solution of what linear system? Give your answer in matrix form.

Solution.

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 3 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}.$$

□

(i) [3 points] What is the dimension of $M_{m\times n}$?

Solution. It is $m \cdot n$. □

(j) [3 points] Give the standard basis of P_3.

Solution. $\{1, x, x^2, x^3\}$. □

(k) [3 points] Let $S = \{\mathbf{v}_1, \mathbf{v}_2\}$ be an orthogonal, *but not orthonormal*, basis of a subspace W of V, and $\mathbf{v} \in V$, give the formula for $\operatorname{proj}_W \mathbf{v}$.

Solution. $\operatorname{proj}_W \mathbf{v} = \left(\frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} \cdot \mathbf{v}\right) \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} + \left(\frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} \cdot \mathbf{v}\right) \frac{\mathbf{v}_2}{\|\mathbf{v}_1\|}.$

□

(l) [3 points] If A is a 5 by 4 matrix of rank 3, give the nullities of A and A^{T}.

Solution. Remember: rank plus nullity of A is the number of columns, so the nullity of A is 1. Also, rank plus nullity of A^{T} is the number of columns of A^{T}, which is the number of rows of A. Hence, nullity of A^{T} is 2.

□

(m) [3 points] What condition on the size of the matrix A guarantee that the system $A\mathbf{x} = \mathbf{0}$ has a non-trivial solution? [If there is no such condition, just say so.]

Solution. We need more variables than equations, so A must have more columns than rows. [So, if A is m by n, then we need $n > m$.]

□

4) [15 points] Let

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

(a) [3 points] Find the eigenvalues of A. [You do *not* need to justify this one.]

Solution. Since the matrix is upper triangular, the eigenvalues are the elements in the main diagonal: 1 and -2. □

(b) [6 points] Find the eigenspaces associated to each eigenvalue.

Solution. For $\lambda = 1$ we have:

$$1 \cdot I_3 - A = \begin{bmatrix} 0 & -1 & -3 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

So, we have that the eigenspace associated to 1 is $\mathrm{span}(\{(1,0,0)\})$.

For $\lambda = -2$, we have:

$$-2 \cdot I_3 - A = \begin{bmatrix} -3 & -1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1/3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

So, we have that the eigenspace associated to -2 is $\mathrm{span}(\{(-1/3,1,0),(-1,0,1)\})$.

□

(c) [6 points] Is A diagonalizable? If so, give P such that $P^{-1}AP$ is diagonal *and* the resulting diagonal form. [You do *not* need to justify in this case.] If not, explain why not.

Solution. Yes, since the dimensions of the eigenspaces add up to the number of rows. Then,

$$P = \begin{bmatrix} 1 & -1/3 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

□

5) [20 points] Let

$$\begin{aligned}
\mathbf{v}_1 &= (4, -4, 2, 2, 4, 1, 17),\\
\mathbf{v}_2 &= (-1, 1, -1, 1, -1, -1, -6),\\
\mathbf{v}_3 &= (3, -3, 2, 0, 3, 1, 14),\\
\mathbf{v}_4 &= (10, -10, 5, 5, 10, 3, 43),\\
\mathbf{v}_5 &= (2, -2, 1, 1, 2, 1, 9),
\end{aligned}$$

and $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$, and $V = \operatorname{span}(S)$. Given that

$$\begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \\ \mathbf{v}_4 \\ \mathbf{v}_5 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 2 & 1 & 0 & 3 \\ 0 & 0 & 1 & -3 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{v}_5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 3 & 1 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

answer the questions below. [You do *not* need to justify any of the items below.]

(a) [5 points] What are the dimension of V and $V^{\perp}$ [the orthogonal complement of V in $\mathbb{R}^7$]?

Solution. The dimension of V is the number of leading ones in either matrix in echelon form above, so it is 3.

The dimension of $V^{\perp}$ is the number of columns without leading ones in the *first* matrix, so it is 4.

□

(b) [5 points] Find a basis for V.

Solution. We can take either the first three rows of the first matrix in echelon form, or $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ [using the second matrix in echelon form]. □

(c) [4 points] Find a basis of $V^\perp$.

Solution. The basis can be found by finding a basis for the nullspace of the first matrix:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} r - 2s - t - 3u \\ r \\ 3s - 2u \\ s \\ t \\ -u \\ u \end{bmatrix} = r \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 0 \\ 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + u \begin{bmatrix} -3 \\ 0 \\ -2 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

The column vectors in evidence above form the desired basis. □

(d) [5 points] If possible, find a non-trivial linear combination [i.e., not all coefficients equal to zero] of the elements of S which give the zero vector of $\mathbb{R}^7$. [**Hint:** Start by writing a vector of S as a linear combination of the others.]

Solution. Using the second matrix in echelon form, and denoting its columns by $\mathbf{c}_1$ to $\mathbf{c}_5$, we can easily see that $\mathbf{c}_5 = \mathbf{c}_1 - \mathbf{c}_2 - \mathbf{c}_3$. Thus, $\mathbf{v}_5 = \mathbf{v}_1 - \mathbf{v}_2 - \mathbf{v}3$. Thus,

$$1 \cdot \mathbf{v}_1 + (-1) \cdot \mathbf{v}_2 + (-1) \cdot \mathbf{v}_3 + 0 \cdot \mathbf{v}_4 + (-1) \cdot \mathbf{v}_5 = \mathbf{0}$$

□

(e) [5 points] Which vectors from the standard basis of $\mathbb{R}^7$ you can add to the vectors in the basis of V you've found above to obtain a basis of all of $\mathbb{R}^7$?

Solution. We just add standard basis vectors with leading ones in columns which have no leading ones in the first matrix in echelon form. So, we add $\{\mathbf{e}_2, \mathbf{e}_4, \mathbf{e}_5, \mathbf{e}_7\}$. □