Skills

- Determine whether a given equation is linear.
- Determine whether a given n-tuple is a solution of a linear system.
- Find the augmented matrix of a linear system.
- Find the linear system corresponding to a given augmented matrix.
- Perform elementary row operations on a linear system and on its corresponding augmented matrix.
- Determine whether a linear system is consistent or inconsistent.
- Find the set of solutions to a consistent linear system.

Exercise Set 1.1

1. In each part, determine whether the equation is linear in x_1, x_2, and x_3.

(a) $x_1 + 5x_2 - \sqrt{2}x_3 = 1$ (b) $x_1 + 3x_2 + x_1x_3 = 2$

(c) $x_1 = -7x_2 + 3x_3$ (d) $x_1^{-2} + x_2 + 8x_3 = 5$

(e) $x_1^{3/5} - 2x_2 + x_3 = 4$

(f) $\pi x_1 - \sqrt{2}x_2 + \frac{1}{3}x_3 = 7^{1/3}$

2. In each part, determine whether the equations form a linear system.

(a) $\begin{aligned} -2x + 4y + z &= 2 \\ 3x - \frac{2}{y} &= 0 \end{aligned}$ (b) $\begin{aligned} x &= 4 \\ 2x &= 8 \end{aligned}$

(c) $\begin{aligned} 4x - y + 2z &= -1 \\ -x + (\ln 2)y - 3z &= 0 \end{aligned}$

(d) $\begin{aligned} 3z + x &= -4 \\ y + 5z &= 1 \\ 6x + 2z &= 3 \\ -x - y - z &= 4 \end{aligned}$

3. In each part, determine whether the equations form a linear system.

(a) $\begin{aligned} 2x_1 \qquad\qquad\quad - x_4 &= 5 \\ -x_1 + 5x_2 + 3x_3 - 2x_4 &= -1 \end{aligned}$

(b) $\begin{aligned} \sin(2x_1 + x_3) &= \sqrt{5} \\ e^{-2x_2 - 2x_4} &= \frac{1}{x_2} \\ 4x_4 &= 4 \end{aligned}$

(c) $\begin{aligned} 7x_1 - x_2 + 2x_3 &= 0 \\ 2x_1 + x_2 - x_3x_4 &= 3 \\ -x_1 + 5x_2 - x_4 &= -1 \end{aligned}$ (d) $x_1 + x_2 = x_3 + x_4$

4. For each system in Exercise 2 that is linear, determine whether it is consistent.

5. For each system in Exercise 3 that is linear, determine whether it is consistent.

6. Write a system of linear equations consisting of three equations in three unknowns with

(a) no solutions.

(b) exactly one solution.

(c) infinitely many solutions.

7. In each part, determine whether the given vector is a solution of the linear system

$$\begin{aligned} 2x_1 - 4x_2 - x_3 &= 1 \\ x_1 - 3x_2 + x_3 &= 1 \\ 3x_1 - 5x_2 - 3x_3 &= 1 \end{aligned}$$

(a) $(3, 1, 1)$ (b) $(3, -1, 1)$ (c) $(13, 5, 2)$

(d) $(\frac{13}{2}, \frac{5}{2}, 2)$ (e) $(17, 7, 5)$

8. In each part, determine whether the given vector is a solution of the linear system

$$\begin{aligned} x_1 + 2x_2 - 2x_3 &= 3 \\ 3x_1 - x_2 + x_3 &= 1 \\ -x_1 + 5x_2 - 5x_3 &= 5 \end{aligned}$$

(a) $(\frac{5}{7}, \frac{8}{7}, 1)$ (b) $(\frac{5}{7}, \frac{8}{7}, 0)$ (c) $(5, 8, 1)$

(d) $(\frac{5}{7}, \frac{10}{7}, \frac{2}{7})$ (e) $(\frac{5}{7}, \frac{22}{7}, 2)$

9. In each part, find the solution set of the linear equation by using parameters as necessary.

(a) $7x - 5y = 3$

(b) $-8x_1 + 2x_2 - 5x_3 + 6x_4 = 1$

10. In each part, find the solution set of the linear equation by using parameters as necessary.

(a) $3x_1 - 5x_2 + 4x_3 = 7$

(b) $3v - 8w + 2x - y + 4z = 0$

11. In each part, find a system of linear equations corresponding to the given augmented matrix.

(a) $\begin{bmatrix} 2 & 0 & 0 \\ 3 & -4 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 0 & -2 & 5 \\ 7 & 1 & 4 & -3 \\ 0 & -2 & 1 & 7 \end{bmatrix}$

(c) $\begin{bmatrix} 7 & 2 & 1 & -3 & 5 \\ 1 & 2 & 4 & 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 0 & 0 & 7 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$

12. In each part, find a system of linear equations corresponding to the given augmented matrix.

(a) $\begin{bmatrix} 2 & -1 \\ -4 & -6 \\ 1 & -1 \\ 3 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 3 & -1 & -1 & -1 \\ 5 & 2 & 0 & -3 & -6 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ -4 & -3 & -2 & -1 \\ 5 & -6 & 1 & 1 \\ -8 & 0 & 0 & 3 \end{bmatrix}$

(d) $\begin{bmatrix} 3 & 0 & 1 & -4 & 3 \\ -4 & 0 & 4 & 1 & -3 \\ -1 & 3 & 0 & -2 & -9 \\ 0 & 0 & 0 & -1 & -2 \end{bmatrix}$

13. In each part, find the augmented matrix for the given system of linear equations.

(a) $\begin{aligned} -2x_1 &= 6 \\ 3x_1 &= 8 \\ 9x_1 &= -3 \end{aligned}$ (b) $\begin{aligned} 6x_1 - x_2 + 3x_3 &= 4 \\ 5x_2 - x_3 &= 1 \end{aligned}$

(c) $\begin{aligned} 2x_2 - 3x_4 + x_5 &= 0 \\ -3x_1 - x_2 + x_3 &= -1 \\ 6x_1 + 2x_2 - x_3 + 2x_4 - 3x_5 &= 6 \end{aligned}$

(d) $x_1 - x_5 = 7$

14. In each part, find the augmented matrix for the given system of linear equations.

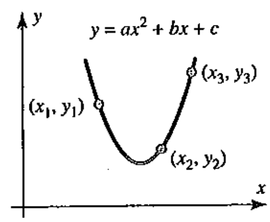
(a) $\begin{aligned} 3x_1 - 2x_2 &= -1 \\ 4x_1 + 5x_2 &= 3 \\ 7x_1 + 3x_2 &= 2 \end{aligned}$ (b) $\begin{aligned} 2x_1 + 2x_3 &= 1 \\ 3x_1 - x_2 + 4x_3 &= 7 \\ 6x_1 + x_2 - x_3 &= 0 \end{aligned}$

(c) $\begin{aligned} x_1 + 2x_2 - x_4 + x_5 &= 1 \\ 3x_2 + x_3 - x_5 &= 2 \\ x_3 + 7x_4 &= 1 \end{aligned}$

(d) $\begin{aligned} x_1 &= 1 \\ x_2 &= 2 \\ x_3 &= 3 \end{aligned}$

15. The curve $y = ax^2 + bx + c$ shown in the accompanying figure passes through the points (x_1, y_1), (x_2, y_2), and (x_3, y_3). Show that the coefficients a, b, and c are a solution of the system of linear equations whose augmented matrix is14

$$\begin{bmatrix} x_1^2 & x_1 & 1 & y_1 \\ x_2^2 & x_2 & 1 & y_2 \\ x_3^2 & x_3 & 1 & y_3 \end{bmatrix}$$



◀ **Figure Ex-15**

16. Explain why each of the three elementary row operations does not affect the solution set of a linear system.

17. Show that if the linear equations

$$x_1 + kx_2 = c \quad \text{and} \quad x_1 + lx_2 = d$$

have the same solution set, then the two equations are identical (i.e., $k = l$ and $c = d$).

True-False Exercises

In parts (a)–(h) determine whether the statement is true or false, and justify your answer.

(a) A linear system whose equations are all homogeneous must be consistent.

(b) Multiplying a linear equation through by zero is an acceptable elementary row operation.

(c) The linear system

$$\begin{aligned} x - y &= 3 \\ 2x - 2y &= k \end{aligned}$$

cannot have a unique solution, regardless of the value of k.

(d) A single linear equation with two or more unknowns must always have infinitely many solutions.

(e) If the number of equations in a linear system exceeds the number of unknowns, then the system must be inconsistent.

(f) If each equation in a consistent linear system is multiplied through by a constant c, then all solutions to the new system can be obtained by multiplying solutions from the original system by c.

(g) Elementary row operations permit one equation in a linear system to be subtracted from another.

(h) The linear system with corresponding augmented matrix

$$\begin{bmatrix} 2 & -1 & 4 \\ 0 & 0 & -1 \end{bmatrix}$$

is consistent.

so unless precautions are taken, successive calculations may degrade an answer to a degree that makes it useless. Algorithms (procedures) in which this happens are called ***unstable***. There are various techniques for minimizing roundoff error and instability. For example, it can be shown that for large linear systems Gauss–Jordan elimination involves roughly 50% more operations than Gaussian elimination, so most computer algorithms are based on the latter method. Some of these matters will be considered in Chapter 9.

Concept Review

- Reduced row echelon form
- Row echelon form
- Leading 1
- Leading variables
- Free variables
- General solution to a linear system
- Gaussian elimination
- Gauss–Jordan elimination
- Forward phase
- Backward phase
- Homogeneous linear system
- Trivial solution
- Nontrivial solution
- Dimension Theorem for Homogeneous Systems
- Back-substitution

Skills

- Recognize whether a given matrix is in row echelon form, reduced row echelon form, or neither.
- Construct solutions to linear systems whose corresponding augmented matrices that are in row echelon form or reduced row echelon form.
- Use Gaussian elimination to find the general solution of a linear system.
- Use Gauss–Jordan elimination to find the general solution of a linear system.
- Analyze homogeneous linear systems using the Free Variable Theorem for Homogeneous Systems.

Exercise Set 1.2

1. In each part, determine whether the matrix is in row echelon form, reduced row echelon form, both, or neither.

(a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 2 & 4 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(f) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ (g) $\begin{bmatrix} 1 & -7 & 5 & 5 \\ 0 & 1 & 3 & 2 \end{bmatrix}$

2. In each part, determine whether the matrix is in row echelon form, reduced row echelon form, both, or neither.

(a) $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 3 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 5 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(f) $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 0 & 7 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ (g) $\begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$

3. In each part, suppose that the augmented matrix for a system of linear equations has been reduced by row operations to the given row echelon form. Solve the system.

(a) $\begin{bmatrix} 1 & -3 & 4 & 7 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 8 & -5 & 6 \\ 0 & 1 & 4 & -9 & 3 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 7 & -2 & 0 & -8 & -3 \\ 0 & 0 & 1 & 1 & 6 & 5 \\ 0 & 0 & 0 & 1 & 3 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & -3 & 7 & 1 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

4. In each part, suppose that the augmented matrix for a system of linear equations has been reduced by row operations to the given row echelon form. Solve the system.

(a) $\begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 7 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 0 & -7 & 8 \\ 0 & 1 & 0 & 3 & 2 \\ 0 & 0 & 1 & 1 & -5 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & -6 & 0 & 0 & 3 & -2 \\ 0 & 0 & 1 & 0 & 4 & 7 \\ 0 & 0 & 0 & 1 & 5 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

In Exercises 5–8, solve the linear system by Gauss–Jordan elimination.

5. $\begin{aligned} x_1 + x_2 + 2x_3 &= 8 \\ -x_1 - 2x_2 + 3x_3 &= 1 \\ 3x_1 - 7x_2 + 4x_3 &= 10 \end{aligned}$

6. $\begin{aligned} 2x_1 + 2x_2 + 2x_3 &= 0 \\ -2x_1 + 5x_2 + 2x_3 &= 1 \\ 8x_1 + x_2 + 4x_3 &= -1 \end{aligned}$

7. $\begin{aligned} x - y + 2z - w &= -1 \\ 2x + y - 2z - 2w &= -2 \\ -x + 2y - 4z + w &= 1 \\ 3x - 3w &= -3 \end{aligned}$

8. $\begin{aligned} -2b + 3c &= 1 \\ 3a + 6b - 3c &= -2 \\ 6a + 6b + 3c &= 5 \end{aligned}$

In Exercises 9–12, solve the linear system by Gaussian elimination.

9. Exercise 5

10. Exercise 6

11. Exercise 7

12. Exercise 8

In Exercises 13–16, determine whether the homogeneous system has nontrivial solutions by inspection (without pencil and paper).

13. $\begin{aligned} 2x_1 - 3x_2 + 4x_3 - x_4 &= 0 \\ 7x_1 + x_2 - 8x_3 + 9x_4 &= 0 \\ 2x_1 + 8x_2 + x_3 - x_4 &= 0 \end{aligned}$

14. $\begin{aligned} x_1 + 3x_2 - x_3 &= 0 \\ x_2 - 8x_3 &= 0 \\ 4x_3 &= 0 \end{aligned}$

15. $\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= 0 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= 0 \end{aligned}$

16. $\begin{aligned} 3x_1 - 2x_2 &= 0 \\ 6x_1 - 4x_2 &= 0 \end{aligned}$

In Exercises 17–24, solve the given linear system by any method.

17. $\begin{aligned} 2x_1 + x_2 + 3x_3 &= 0 \\ x_1 + 2x_2 &= 0 \\ x_2 + x_3 &= 0 \end{aligned}$

18. $\begin{aligned} 2x - y - 3z &= 0 \\ -x + 2y - 3z &= 0 \\ x + y + 4z &= 0 \end{aligned}$

19. $\begin{aligned} 3x_1 + x_2 + x_3 + x_4 &= 0 \\ 5x_1 - x_2 + x_3 - x_4 &= 0 \end{aligned}$

20. $\begin{aligned} v + 3w - 2x &= 0 \\ 2u + v - 4w + 3x &= 0 \\ 2u + 3v + 2w - x &= 0 \\ -4u - 3v + 5w - 4x &= 0 \end{aligned}$

21. $\begin{aligned} 2x + 2y + 4z &= 0 \\ w - y - 3z &= 0 \\ 2w + 3x + y + z &= 0 \\ -2w + x + 3y - 2z &= 0 \end{aligned}$

22. $\begin{aligned} x_1 + 3x_2 + x_4 &= 0 \\ x_1 + 4x_2 + 2x_3 &= 0 \\ -2x_2 - 2x_3 - x_4 &= 0 \\ 2x_1 - 4x_2 + x_3 + x_4 &= 0 \\ x_1 - 2x_2 - x_3 + x_4 &= 0 \end{aligned}$

23. $\begin{aligned} 2I_1 - I_2 + 3I_3 + 4I_4 &= 9 \\ I_1 - 2I_3 + 7I_4 &= 11 \\ 3I_1 - 3I_2 + I_3 + 5I_4 &= 8 \\ 2I_1 + I_2 + 4I_3 + 4I_4 &= 10 \end{aligned}$

24. $\begin{aligned} Z_3 + Z_4 + Z_5 &= 0 \\ -Z_1 - Z_2 + 2Z_3 - 3Z_4 + Z_5 &= 0 \\ Z_1 + Z_2 - 2Z_3 - Z_5 &= 0 \\ 2Z_1 + 2Z_2 - Z_3 + Z_5 &= 0 \end{aligned}$

In Exercises 25–28, determine the values of a for which the system has no solutions, exactly one solution, or infinitely many solutions.

25. $\begin{aligned} x + 2y - 3z &= 4 \\ 3x - y + 5z &= 2 \\ 4x + y + (a^2 - 14)z &= a + 2 \end{aligned}$

26. $\begin{aligned} x + 2y + z &= 2 \\ 2x - 2y + 3z &= 1 \\ x + 2y - (a^2 - 3)z &= a \end{aligned}$

27. $\begin{aligned} x + 2y &= 1 \\ 2x + (a^2 - 5)y &= a - 1 \end{aligned}$

28. $\begin{aligned} x + y + 7z &= -7 \\ 2x + 3y + 17z &= -16 \\ x + 2y + (a^2 + 1)z &= 3a \end{aligned}$

In Exercises 29–30, solve the following systems, where a, b, and c are constants.

29. $$\begin{aligned} 2x + y &= a \\ 3x + 6y &= b \end{aligned}$$

30. $$\begin{aligned} x_1 + x_2 + x_3 &= a \\ 2x_1 + 2x_3 &= b \\ 3x_2 + 3x_3 &= c \end{aligned}$$

31. Find two different row echelon forms of

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$$

This exercise shows that a matrix can have multiple row echelon forms.

32. Reduce

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & -2 & -29 \\ 3 & 4 & 5 \end{bmatrix}$$

to reduced row echelon form without introducing fractions at any intermediate stage.

33. Show that the following nonlinear system has 18 solutions if $0 \le \alpha \le 2\pi$, $0 \le \beta \le 2\pi$, and $0 \le \gamma < 2\pi$.

$$\begin{aligned} \sin\alpha + 2\cos\beta + 3\tan\gamma &= 0 \\ 2\sin\alpha + 5\cos\beta + 3\tan\gamma &= 0 \\ -\sin\alpha - 5\cos\beta + 5\tan\gamma &= 0 \end{aligned}$$

[*Hint:* Begin by making the substitutions $x = \sin\alpha$, $y = \cos\beta$, and $z = \tan\gamma$.]

34. Solve the following system of nonlinear equations for the unknown angles α, β, and γ, where $0 \le \alpha \le 2\pi$, $0 \le \beta \le 2\pi$, and $0 \le \gamma < \pi$.

$$\begin{aligned} 2\sin\alpha - \cos\beta + 3\tan\gamma &= 3 \\ 4\sin\alpha + 2\cos\beta - 2\tan\gamma &= 2 \\ 6\sin\alpha - 3\cos\beta + \tan\gamma &= 9 \end{aligned}$$

35. Solve the following system of nonlinear equations for x, y, and z.

$$\begin{aligned} x^2 + y^2 + z^2 &= 6 \\ x^2 - y^2 + 2z^2 &= 2 \\ 2x^2 + y^2 - z^2 &= 3 \end{aligned}$$

[*Hint:* Begin by making the substitutions $X = x^2$, $Y = y^2$, $Z = z^2$.]

36. Solve the following system for x, y, and z.

$$\begin{aligned} \frac{1}{x} + \frac{2}{y} - \frac{4}{z} &= 1 \\ \frac{2}{x} + \frac{3}{y} + \frac{8}{z} &= 0 \\ -\frac{1}{x} + \frac{9}{y} + \frac{10}{z} &= 5 \end{aligned}$$

37. Find the coefficients a, b, c, and d so that the curve shown in the accompanying figure is the graph of the equation $y = ax^3 + bx^2 + cx + d$.

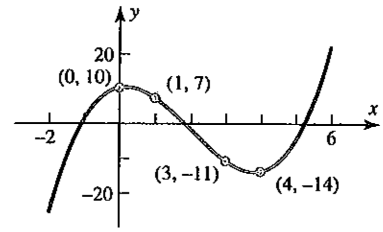


Figure Ex-37

38. Find the coefficients a, b, c, and d so that the curve shown in the accompanying figure is given by the equation $ax^2 + ay^2 + bx + cy + d = 0$.

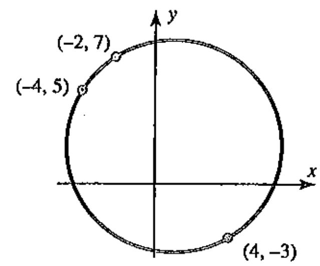


Figure Ex-38

39. If the linear system

$$\begin{aligned} a_1x + b_1y + c_1z &= 0 \\ a_2x - b_2y + c_2z &= 0 \\ a_3x + b_3y - c_3z &= 0 \end{aligned}$$

has only the trivial solution, what can be said about the solutions of the following system?

$$\begin{aligned} a_1x + b_1y + c_1z &= 3 \\ a_2x - b_2y + c_2z &= 7 \\ a_3x + b_3y - c_3z &= 11 \end{aligned}$$

40. (a) If A is a 3×5 matrix, then what is the maximum possible number of leading 1's in its reduced row echelon form?

(b) If B is a 3×6 matrix whose last column has all zeros, then what is the maximum possible number of parameters in the general solution of the linear system with augmented matrix B?

(c) If C is a 5×3 matrix, then what is the minimum possible number of rows of zeros in any row echelon form of C?

41. (a) Prove that if $ad - bc \ne 0$, then the reduced row echelon form of

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ is } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(b) Use the result in part (a) to prove that if $ad - bc \ne 0$, then the linear system

$$\begin{aligned} ax + by &= k \\ cx + dy &= l \end{aligned}$$

has exactly one solution.

42. Consider the system of equations

$$\begin{aligned} ax + by &= 0 \\ cx + dy &= 0 \\ ex + fy &= 0 \end{aligned}$$

Discuss the relative positions of the lines $ax + by = 0$, $cx + dy = 0$, and $ex + fy = 0$ when (a) the system has only the trivial solution, and (b) the system has nontrivial solutions.

43. Describe all possible reduced row echelon forms of

(a) $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ (b) $\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & p & q \end{bmatrix}$

True-False Exercises

In parts (a)–(i) determine whether the statement is true or false, and justify your answer.

(a) If a matrix is in reduced row echelon form, then it is also in row echelon form.

(b) If an elementary row operation is applied to a matrix that is in row echelon form, the resulting matrix will still be in row echelon form.

(c) Every matrix has a unique row echelon form.

(d) A homogeneous linear system in n unknowns whose corresponding augmented matrix has a reduced row echelon form with r leading 1's has $n - r$ free variables.

(e) All leading 1's in a matrix in row echelon form must occur in different columns.

(f) If every column of a matrix in row echelon form has a leading 1 then all entries that are not leading 1's are zero.

(g) If a homogeneous linear system of n equations in n unknowns has a corresponding augmented matrix with a reduced row echelon form containing n leading 1's, then the linear system has only the trivial solution.

(h) If the reduced row echelon form of the augmented matrix for a linear system has a row of zeros, then the system must have infinitely many solutions.

(i) If a linear system has more unknowns than equations, then it must have infinitely many solutions.

1.3 Matrices and Matrix Operations

Rectangular arrays of real numbers arise in contexts other than as augmented matrices for linear systems. In this section we will begin to study matrices as objects in their own right by defining operations of addition, subtraction, and multiplication on them.

Matrix Notation and Terminology

In Section 1.2 we used rectangular arrays of numbers, called *augmented matrices*, to abbreviate systems of linear equations. However, rectangular arrays of numbers occur in other contexts as well. For example, the following rectangular array with three rows and seven columns might describe the number of hours that a student spent studying three subjects during a certain week:

	Mon.	Tues.	Wed.	Thurs.	Fri.	Sat.	Sun.
Math	2	3	2	4	1	4	2
History	0	3	1	4	3	2	2
Language	4	1	3	1	0	0	2

If we suppress the headings, then we are left with the following rectangular array of numbers with three rows and seven columns, called a "matrix":

$$\begin{bmatrix} 2 & 3 & 2 & 4 & 1 & 4 & 2 \\ 0 & 3 & 1 & 4 & 3 & 2 & 2 \\ 4 & 1 & 3 & 1 & 0 & 0 & 2 \end{bmatrix}$$

More generally, we make the following definition.

DEFINITION 8 If A is a square matrix, then the ***trace of A***, denoted by $\text{tr}(A)$, is defined to be the sum of the entries on the main diagonal of A. The trace of A is undefined if A is not a square matrix.

▷ EXAMPLE 11 Trace of a Matrix

The following are examples of matrices and their traces.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 2 & 7 & 0 \\ 3 & 5 & -8 & 4 \\ 1 & 2 & 7 & -3 \\ 4 & -2 & 1 & 0 \end{bmatrix}$$

$$\text{tr}(A) = a_{11} + a_{22} + a_{33} \qquad \text{tr}(B) = -1 + 5 + 7 + 0 = 11$$ ◀

In the exercises you will have some practice working with the transpose and trace operations.

Concept Review

- Matrix
- Entries
- Column vector (or column matrix)
- Row vector (or row matrix)
- Square matrix
- Main diagonal
- Equal matrices
- Matrix operations: sum, difference, scalar multiplication
- Linear combination of matrices
- Product of matrices (matrix multiplication)
- Partitioned matrices
- Submatrices
- Row-column method
- Column method
- Row method
- Coefficient matrix of a linear system
- Transpose
- Trace

Skills

- Determine the size of a given matrix.
- Identify the row vectors and column vectors of a given matrix.
- Perform the arithmetic operations of matrix addition, subtraction, scalar multiplication, and multiplication.
- Determine whether the product of two given matrices is defined.
- Compute matrix products using the row-column method, the column method, and the row method.
- Express the product of a matrix and a column vector as a linear combination of the columns of the matrix.
- Express a linear system as a matrix equation, and identify the coefficient matrix.
- Compute the transpose of a matrix.
- Compute the trace of a square matrix.

Exercise Set 1.3

1. Suppose that A, B, C, D, and E are matrices with the following sizes:

A	B	C	D	E
(4×5)	(4×5)	(5×2)	(4×2)	(5×4)

In each part, determine whether the given matrix expression is defined. For those that are defined, give the size of the resulting matrix.

(a) BA (b) $AC + D$ (c) $AE + B$

(d) $AB + B$ (e) $E(A + B)$ (f) $E(AC)$

(g) E^TA (h) $(A^T + E)D$

2. Suppose that A, B, C, D, and E are matrices with the following sizes:

A	B	C	D	E
(3×1)	(3×6)	(6×2)	(2×6)	(1×3)

In each part, determine whether the given matrix expression is defined. For those that are defined, give the size of the resulting matrix.

(a) EA (b) AB^T (c) $B^T(A+E^T)$

(d) $2A+C$ (e) $(C^T+D)B^T$ (f) $CD+B^TE^T$

(g) $(BD^T)C^T$ (h) $DC+EA$

3. Consider the matrices

$$A=\begin{bmatrix}3&0\\-1&2\\1&1\end{bmatrix},\quad B=\begin{bmatrix}4&-1\\0&2\end{bmatrix},\quad C=\begin{bmatrix}1&4&2\\3&1&5\end{bmatrix},$$

$$D=\begin{bmatrix}1&5&2\\-1&0&1\\3&2&4\end{bmatrix},\quad E=\begin{bmatrix}6&1&3\\-1&1&2\\4&1&3\end{bmatrix}$$

In each part, compute the given expression (where possible).

(a) $D+E$ (b) $D-E$ (c) $5A$

(d) $-7C$ (e) $2B-C$ (f) $4E-2D$

(g) $-3(D+2E)$ (h) $A-A$ (i) $\operatorname{tr}(D)$

(j) $\operatorname{tr}(D-3E)$ (k) $4\operatorname{tr}(7B)$ (l) $\operatorname{tr}(A)$

4. Using the matrices in Exercise 3, in each part compute the given expression (where possible).

(a) $2A^T+C$ (b) D^T-E^T (c) $(D-E)^T$

(d) B^T+5C^T (e) $\frac{1}{2}C^T-\frac{1}{4}A$ (f) $B-B^T$

(g) $2E^T-3D^T$ (h) $(2E^T-3D^T)^T$ (i) $(CD)E$

(j) $C(BA)$ (k) $\operatorname{tr}(DE^T)$ (l) $\operatorname{tr}(BC)$

5. Using the matrices in Exercise 3, in each part compute the given expression (where possible).

(a) AB (b) BA (c) $(3E)D$

(d) $(AB)C$ (e) $A(BC)$ (f) CC^T

(g) $(DA)^T$ (h) $(C^TB)A^T$ (i) $\operatorname{tr}(DD^T)$

(j) $\operatorname{tr}(4E^T-D)$ (k) $\operatorname{tr}(C^TA^T+2E^T)$ (l) $\operatorname{tr}((EC^T)^TA)$

6. Using the matrices in Exercise 3, in each part compute the given expression (where possible).

(a) $(2D^T-E)A$ (b) $(4B)C+2B$

(c) $(-AC)^T+5D^T$ (d) $(BA^T-2C)^T$

(e) $B^T(CC^T-A^TA)$ (f) $D^TE^T-(ED)^T$

7. Let

$$A=\begin{bmatrix}3&-2&7\\6&5&4\\0&4&9\end{bmatrix}\quad\text{and}\quad B=\begin{bmatrix}6&-2&4\\0&1&3\\7&7&5\end{bmatrix}$$

Use the row method or column method (as appropriate) to find

(a) the first row of AB. (b) the third row of AB.

(c) the second column of AB. (d) the first column of BA.

(e) the third row of AA. (f) the third column of AA.

8. Referring to the matrices in Exercise 7, use the row method or column method (as appropriate) to find

(a) the first column of AB. (b) the third column of BB.

(c) the second row of BB. (d) the first column of AA.

(e) the third column of AB. (f) the first row of BA.

9. Referring to the matrices in Exercise 7 and Example 9,

(a) express each column vector of AA as a linear combination of the column vectors of A.

(b) express each column vector of BB as a linear combination of the column vectors of B.

10. Referring to the matrices in Exercise 7 and Example 9,

(a) express each column vector of AB as a linear combination of the column vectors of A.

(b) express each column vector of BA as a linear combination of the column vectors of B.

11. In each part, find matrices A, $\mathbf{x}$, and $\mathbf{b}$ that express the given system of linear equations as a single matrix equation $A\mathbf{x}=\mathbf{b}$, and write out this matrix equation.

(a)
$$\begin{aligned}2x_1-3x_2+5x_3&=7\\9x_1-x_2+x_3&=-1\\x_1+5x_2+4x_3&=0\end{aligned}$$

(b)
$$\begin{aligned}4x_1\qquad\;\, -3x_3+x_4&=1\\5x_1+x_2\qquad\;\, -8x_4&=3\\2x_1-5x_2+9x_3-x_4&=0\\3x_2-x_3+7x_4&=2\end{aligned}$$

12. In each part, find matrices A, $\mathbf{x}$, and $\mathbf{b}$ that express the given system of linear equations as a single matrix equation $A\mathbf{x}=\mathbf{b}$, and write out this matrix equation.

(a)
$$\begin{aligned}x_1-2x_2+3x_3&=-3\\2x_1+x_2\qquad&=0\\-3x_2+4x_3&=1\\x_1\qquad+x_3&=5\end{aligned}$$

(b)
$$\begin{aligned}3x_1+3x_2+3x_3&=-3\\-x_1-5x_2-2x_3&=3\\-4x_2+x_3&=0\end{aligned}$$

13. In each part, express the matrix equation as a system of linear equations.

(a) $$\begin{bmatrix}5&6&-7\\-1&-2&3\\0&4&-1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}2\\0\\3\end{bmatrix}$$

(b) $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 0 \\ 5 & -3 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -9 \end{bmatrix}$

14. In each part, express the matrix equation as a system of linear equations.

(a) $\begin{bmatrix} 3 & -1 & 2 \\ 4 & 3 & 7 \\ -2 & 1 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$

(b) $\begin{bmatrix} 3 & -2 & 0 & 1 \\ 5 & 0 & 2 & -2 \\ 3 & 1 & 4 & 7 \\ -2 & 5 & 1 & 6 \end{bmatrix}\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

▶ In Exercises **15–16**, find all values of k, if any, that satisfy the equation. ◀

15. $\begin{bmatrix} k & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & -3 \end{bmatrix}\begin{bmatrix} k \\ 1 \\ 1 \end{bmatrix} = 0$

16. $\begin{bmatrix} 2 & 2 & k \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 3 \\ 0 & 3 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ k \end{bmatrix} = 0$

▶ In Exercises **17–18**, solve the matrix equation for a, b, c, and d.

17. $\begin{bmatrix} a & 3 \\ -1 & a+b \end{bmatrix} = \begin{bmatrix} 4 & d-2c \\ d+2c & -2 \end{bmatrix}$

18. $\begin{bmatrix} a-b & b+a \\ 3d+c & 2d-c \end{bmatrix} = \begin{bmatrix} 8 & 1 \\ 7 & 6 \end{bmatrix}$

19. Let A be any $m \times n$ matrix and let 0 be the $m \times n$ matrix each of whose entries is zero. Show that if $kA = 0$, then $k = 0$ or $A = 0$.

20. (a) Show that if AB and BA are both defined, then AB and BA are square matrices.

(b) Show that if A is an $m \times n$ matrix and $A(BA)$ is defined, then B is an $n \times m$ matrix.

21. Prove: If A and B are $n \times n$ matrices, then

$$\operatorname{tr}(A+B) = \operatorname{tr}(A) + \operatorname{tr}(B)$$

22. (a) Show that if A has a row of zeros and B is any matrix for which AB is defined, then AB also has a row of zeros.

(b) Find a similar result involving a column of zeros.

23. In each part, find a 6×6 matrix $[a_{ij}]$ that satisfies the stated condition. Make your answers as general as possible by using letters rather than specific numbers for the nonzero entries.

(a) $a_{ij} = 0$ if $i \neq j$ (b) $a_{ij} = 0$ if $i > j$

(c) $a_{ij} = 0$ if $i < j$

(d) $a_{ij} = 0$ if $|i-j| > 1$

24. Find the 4×4 matrix $A = [a_{ij}]$ whose entries satisfy the stated condition.

(a) $a_{ij} = i + j$ (b) $a_{ij} = i^{j-1}$

(c) $a_{ij} = \begin{cases} 1 & \text{if } |i-j| > 1 \\ -1 & \text{if } |i-j| \leq 1 \end{cases}$

25. Consider the function $y = f(x)$ defined for 2×1 matrices x by $y = Ax$, where

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Plot $f(x)$ together with x in each case below. How would you describe the action of f?

(a) $x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (b) $x = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

(c) $x = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ (d) $x = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$

26. Let I be the $n \times n$ matrix whose entry in row i and column j is

$$\begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Show that $AI = IA = A$ for every $n \times n$ matrix A.

27. How many 3×3 matrices A can you find such that

$$A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+y \\ x-y \\ 0 \end{bmatrix}$$

for all choices of x, y, and z?

28. How many 3×3 matrices A can you find such that

$$A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} xy \\ 0 \\ 0 \end{bmatrix}$$

for all choices of x, y, and z?

29. A matrix B is said to be a ***square root*** of a matrix A if $BB = A$.

(a) Find two square roots of $A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$.

(b) How many different square roots can you find of $A = \begin{bmatrix} 5 & 0 \\ 0 & 9 \end{bmatrix}$?

(c) Do you think that every 2×2 matrix has at least one square root? Explain your reasoning.

30. Let 0 denote a 2×2 matrix, each of whose entries is zero.

(a) Is there a 2×2 matrix A such that $A \neq 0$ and $AA = 0$? Justify your answer.

(b) Is there a 2×2 matrix A such that $A \neq 0$ and $AA = A$? Justify your answer.

True-False Exercises

In parts (a)–(o) determine whether the statement is true or false, and justify your answer.

(a) The matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ has no main diagonal.

(b) An $m \times n$ matrix has m column vectors and n row vectors.

(c) If A and B are 2×2 matrices, then $AB = BA$.

(d) The ith row vector of a matrix product AB can be computed by multiplying A by the ith row vector of B.

(e) For every matrix A, it is true that $(A^T)^T = A$.

(f) If A and B are square matrices of the same order, then $\text{tr}(AB) = \text{tr}(A)\text{tr}(B)$.

(g) If A and B are square matrices of the same order, then $(AB)^T = A^T B^T$.

(h) For every square matrix A, it is true that $\text{tr}(A^T) = \text{tr}(A)$.

(i) If A is a 6×4 matrix and B is an $m \times n$ matrix such that $B^T A^T$ is a 2×6 matrix, then $m = 4$ and $n = 2$.

(j) If A is an $n \times n$ matrix and c is a scalar, then $\text{tr}(cA) = c\,\text{tr}(A)$.

(k) If A, B, and C are matrices of the same size such that $A - C = B - C$, then $A = B$.

(l) If A, B, and C are square matrices of the same order such that $AC = BC$, then $A = B$.

(m) If $AB + BA$ is defined, then A and B are square matrices of the same size.

(n) If B has a column of zeros, then so does AB if this product is defined.

(o) If B has a column of zeros, then so does BA if this product is defined.

1.4 Inverses; Algebraic Properties of Matrices

In this section we will discuss some of the algebraic properties of matrix operations. We will see that many of the basic rules of arithmetic for real numbers hold for matrices, but we will also see that some do not.

Properties of Matrix Addition and Scalar Multiplication

The following theorem lists the basic algebraic properties of the matrix operations.

THEOREM 1.4.1 Properties of Matrix Arithmetic

Assuming that the sizes of the matrices are such that the indicated operations can be performed, the following rules of matrix arithmetic are valid.

(a) $A + B = B + A$ **(Commutative law for addition)**

(b) $A + (B + C) = (A + B) + C$ **(Associative law for addition)**

(c) $A(BC) = (AB)C$ **(Associative law for multiplication)**

(d) $A(B + C) = AB + AC$ **(Left distributive law)**

(e) $(B + C)A = BA + CA$ **(Right distributive law)**

(f) $A(B - C) = AB - AC$

(g) $(B - C)A = BA - CA$

(h) $a(B + C) = aB + aC$

(i) $a(B - C) = aB - aC$

(j) $(a + b)C = aC + bC$

(k) $(a - b)C = aC - bC$

(l) $a(bC) = (ab)C$

(m) $a(BC) = (aB)C = B(aC)$

To prove any of the equalities in this theorem we must show that the matrix on the left side has the same size as that on the right and that the corresponding entries on the two

Exercise Set 1.4

1. Let

$$A=\begin{bmatrix}2 & -1 & 3\\ 0 & 4 & 5\\ -2 & 1 & 4\end{bmatrix},\quad B=\begin{bmatrix}8 & -3 & -5\\ 0 & 1 & 2\\ 4 & -7 & 6\end{bmatrix},$$

$$C=\begin{bmatrix}0 & -2 & 3\\ 1 & 7 & 4\\ 3 & 5 & 9\end{bmatrix},\quad a=4,\quad b=-7$$

Show that

(a) $A+(B+C)=(A+B)+C$

(b) $(AB)C=A(BC)$ (c) $(a+b)C=aC+bC$

(d) $a(B-C)=aB-aC$

2. Using the matrices and scalars in Exercise 1, verify that

(a) $a(BC)=(aB)C=B(aC)$

(b) $A(B-C)=AB-AC$ (c) $(B+C)A=BA+CA$

(d) $a(bC)=(ab)C$

3. Using the matrices and scalars in Exercise 1, verify that

(a) $(A^T)^T=A$ (b) $(A+B)^T=A^T+B^T$

(c) $(aC)^T=aC^T$ (d) $(AB)^T=B^TA^T$

In Exercises 4–7, use Theorem 1.4.5 to compute the inverses of the following matrices.

4. $A=\begin{bmatrix}3 & 1\\ 5 & 2\end{bmatrix}$ 5. $B=\begin{bmatrix}2 & -3\\ 4 & 4\end{bmatrix}$

6. $C=\begin{bmatrix}6 & 4\\ -2 & -1\end{bmatrix}$ 7. $D=\begin{bmatrix}2 & 0\\ 0 & 3\end{bmatrix}$

8. Find the inverse of

$$\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$$

9. Find the inverse of

$$\begin{bmatrix}\frac{1}{2}(e^x+e^{-x}) & \frac{1}{2}(e^x-e^{-x})\\ \frac{1}{2}(e^x-e^{-x}) & \frac{1}{2}(e^x+e^{-x})\end{bmatrix}$$

10. Use the matrix A in Exercise 4 to verify that $(A^T)^{-1}=(A^{-1})^T$.

11. Use the matrix B in Exercise 5 to verify that $(B^T)^{-1}=(B^{-1})^T$.

12. Use the matrices A and B in Exercises 4 and 5 to verify that $(AB)^{-1}=B^{-1}A^{-1}$.

13. Use the matrices A, B, and C in Exercises 4–6 to verify that $(ABC)^{-1}=C^{-1}B^{-1}A^{-1}$.

In Exercises 14–17, use the given information to find A.

14. $A^{-1}=\begin{bmatrix}2 & -1\\ 3 & 5\end{bmatrix}$ 15. $(7A)^{-1}=\begin{bmatrix}-3 & 7\\ 1 & -2\end{bmatrix}$

16. $(5A^T)^{-1}=\begin{bmatrix}-3 & -1\\ 5 & 2\end{bmatrix}$ 17. $(I+2A)^{-1}=\begin{bmatrix}-1 & 2\\ 4 & 5\end{bmatrix}$

18. Let A be the matrix

$$\begin{bmatrix}2 & 0\\ 4 & 1\end{bmatrix}$$

In each part, compute the given quantity.

(a) A^3 (b) A^{-3} (c) A^2-2A+I

(d) $p(A)$, where $p(x)=x-2$

(e) $p(A)$, where $p(x)=2x^2-x+1$

(f) $p(A)$, where $p(x)=x^3-2x+4$

19. Repeat Exercise 18 for the matrix

$$A=\begin{bmatrix}3 & 1\\ 2 & 1\end{bmatrix}$$

20. Repeat Exercise 18 for the matrix

$$A=\begin{bmatrix}3 & 0 & -1\\ 0 & -2 & 0\\ 5 & 0 & 2\end{bmatrix}$$

21. Repeat Exercise 18 for the matrix

$$A=\begin{bmatrix}3 & 0 & 0\\ 0 & -1 & 3\\ 0 & -3 & -1\end{bmatrix}$$

In Exercises 22–24, let $p_1(x)=x^2-9$, $p_2(x)=x+3$, and $p_3(x)=x-3$. Show that $p_1(A)=p_2(A)p_3(A)$ for the given matrix.

22. The matrix A in Exercise 18.

23. The matrix A in Exercise 21.

24. An arbitrary square matrix A.

25. Show that if $p(x)=x^2-(a+d)x+(ad-bc)$ and

$$A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$$

then $p(A)=0$.

26. Show that if $p(x)=x^2-(a+b+e)x+a(be-cd)$ and

$$A=\begin{bmatrix}a & 0 & 0\\ 0 & b & c\\ 0 & d & e\end{bmatrix}$$

then $p(A)=0$.

27. Consider the matrix

$$A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

where $a_{11}a_{22}\cdots a_{nn} \neq 0$. Show that A is invertible and find its inverse.

28. Show that if a square matrix A satisfies $A^2 - 3A + I = 0$, then $A^{-1} = 3I - A$.

29. (a) Show that a matrix with a row of zeros cannot have an inverse.

(b) Show that a matrix with a column of zeros cannot have an inverse.

30. Assuming that all matrices are $n \times n$ and invertible, solve for D.

$$ABC^TDBA^TC = AB^T$$

31. Assuming that all matrices are $n \times n$ and invertible, solve for D.

$$C^TB^{-1}A^2BAC^{-1}DA^{-2}B^TC^{-2} = C^T$$

32. If A is a square matrix and n is a positive integer, is it true that $(A^n)^T = (A^T)^n$? Justify your answer.

33. Simplify:

$$(AB)^{-1}(AC^{-1})(D^{-1}C^{-1})^{-1}D^{-1}$$

34. Simplify:

$$(AC^{-1})^{-1}(AC^{-1})(AC^{-1})^{-1}AD^{-1}$$

In Exercises 35–37, determine whether A is invertible, and if so, find the inverse. [*Hint:* Solve $AX = I$ for X by equating corresponding entries on the two sides.]

35. $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

36. $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ **37.** $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$

38. Prove Theorem 1.4.2.

In Exercises **39–42**, use the method of Example 8 to find the unique solution of the given linear system.

39. $\begin{aligned} 3x_1 - 2x_2 &= -1 \\ 4x_1 + 5x_2 &= 3 \end{aligned}$ **40.** $\begin{aligned} -x_1 + 5x_2 &= 4 \\ -x_1 - 3x_2 &= 1 \end{aligned}$

41. $\begin{aligned} 6x_1 + x_2 &= 0 \\ 4x_1 - 3x_2 &= -2 \end{aligned}$ **42.** $\begin{aligned} 2x_1 - 2x_2 &= 4 \\ x_1 + 4x_2 &= 4 \end{aligned}$

43. Prove part (a) of Theorem 1.4.1.

44. Prove part (c) of Theorem 1.4.1.

45. Prove part (f) of Theorem 1.4.1.

46. Prove part (b) of Theorem 1.4.2.

47. Prove part (c) of Theorem 1.4.2.

48. Verify Formula (4) in the text by a direct calculation.

49. Prove part (d) of Theorem 1.4.8.

50. Prove part (e) of Theorem 1.4.8.

51. (a) Show that if A is invertible and $AB = AC$, then $B = C$.

(b) Explain why part (a) and Example 3 do not contradict one another.

52. Show that if A is invertible and k is any nonzero scalar, then $(kA)^n = k^nA^n$ for all integer values of n.

53. (a) Show that if A, B, and $A + B$ are invertible matrices with the same size, then

$$A(A^{-1} + B^{-1})B(A + B)^{-1} = I$$

(b) What does the result in part (a) tell you about the matrix $A^{-1} + B^{-1}$?

54. A square matrix A is said to be ***idempotent*** if $A^2 = A$.

(a) Show that if A is idempotent, then so is $I - A$.

(b) Show that if A is idempotent, then $2A - I$ is invertible and is its own inverse.

55. Show that if A is a square matrix such that $A^k = 0$ for some positive integer k, then the matrix A is invertible and

$$(I - A)^{-1} = I + A + A^2 + \cdots + A^{k-1}$$

True-False Exercises

In parts (a)–(k) determine whether the statement is true or false, and justify your answer.

(a) Two $n \times n$ matrices, A and B, are inverses of one another if and only if $AB = BA = 0$.

(b) For all square matrices A and B of the same size, it is true that $(A + B)^2 = A^2 + 2AB + B^2$.

(c) For all square matrices A and B of the same size, it is true that $A^2 - B^2 = (A - B)(A + B)$.

(d) If A and B are invertible matrices of the same size, then AB is invertible and $(AB)^{-1} = A^{-1}B^{-1}$.

(e) If A and B are matrices such that AB is defined, then it is true that $(AB)^T = A^TB^T$.

(f) The matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible if and only if $ad - bc \neq 0$.

(g) If A and B are matrices of the same size and k is a constant, then $(kA + B)^T = kA^T + B^T$.

(h) If A is an invertible matrix, then so is A^T.

(i) If $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_mx^m$ and I is an identity matrix, then $p(I) = a_0 + a_1 + a_2 + \cdots + a_m$.

(j) A square matrix containing a row or column of zeros cannot be invertible.

(k) The sum of two invertible matrices of the same size must be invertible.

1.5 Elementary Matrices and a Method for Finding A^{-1}

In this section we will develop an algorithm for finding the inverse of a matrix, and we will discuss some of the basic properties of invertible matrices.

In Section 1.1 we defined three elementary row operations on a matrix A:

1. Multiply a row by a nonzero constant c.
2. Interchange two rows.
3. Add a constant c times one row to another.

It should be evident that if we let B be the matrix that results from A by performing one of the operations in this list, then the matrix A can be recovered from B by performing the corresponding operation in the following list:

1. Multiply the same row by $1/c$.
2. Interchange the same two rows.
3. If B resulted by adding c times row r_1 of A to row r_2, then add $-c$ times r_2 to row r_1.

It follows that if B is obtained from A by performing a sequence of elementary row operations, then there is a second sequence of elementary row operations, which when applied to B recovers A (Exercise 43). Accordingly, we make the following definition.

DEFINITION 1 Matrices A and B are said to be ***row equivalent*** if either (hence each) can be obtained from the other by a sequence of elementary row operations.

Our next goal is to show how matrix multiplication can be used to carry out an elementary row operation.

DEFINITION 2 An $n \times n$ matrix is called an ***elementary matrix*** if it can be obtained from the $n \times n$ identity matrix I_n by performing a *single* elementary row operation.

EXAMPLE 1 Elementary Matrices and Row Operations

Listed below are four elementary matrices and the operations that produce them.

$$\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Multiply the second row of I_2 by -3.

Interchange the second and fourth rows of I_4.

Add 3 times the third row of I_3 to the first row.

Multiply the first row of I_3 by 1. ◀

Exercise Set 1.5

1. Decide whether each matrix below is an elementary matrix.

(a) $\begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} -5 & 1 \\ 1 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

2. Decide whether each matrix below is an elementary matrix.

(a) $\begin{bmatrix} 1 & 0 \\ 0 & \sqrt{3} \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 9 \\ 0 & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

3. Find a row operation and the corresponding elementary matrix that will restore the given elementary matrix to the identity matrix.

(a) $\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} -7 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

4. Find a row operation and the corresponding elementary matrix that will restore the given elementary matrix to the identity matrix.

(a) $\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 & -\frac{1}{7} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

5. In each part, an elementary matrix E and a matrix A are given. Write down the row operation corresponding to E and show that the product EA results from applying the row operation to A.

(a) $E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -1 & -2 & 5 & -1 \\ 3 & -6 & -6 & -6 \end{bmatrix}$

(b) $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 2 & -1 & 0 & -4 & -4 \\ 1 & -3 & -1 & 5 & 3 \\ 2 & 0 & 1 & 3 & -1 \end{bmatrix}$

(c) $E = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

6. In each part, an elementary matrix E and a matrix A are given. Write down the row operation corresponding to E and show that the product EA results from applying the row operation to A.

(a) $E = \begin{bmatrix} -6 & 0 \\ 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} -1 & -2 & 5 & -1 \\ 3 & -6 & -6 & -6 \end{bmatrix}$

(b) $E = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 2 & -1 & 0 & -4 & -4 \\ 1 & -3 & -1 & 5 & 3 \\ 2 & 0 & 1 & 3 & -1 \end{bmatrix}$

(c) $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

▶ In Exercises 7–8, use the following matrices.

$$A = \begin{bmatrix} 3 & 4 & 1 \\ 2 & -7 & -1 \\ 8 & 1 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 8 & 1 & 5 \\ 2 & -7 & -1 \\ 3 & 4 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 3 & 4 & 1 \\ 2 & -7 & -1 \\ 2 & -7 & 3 \end{bmatrix}, \quad D = \begin{bmatrix} 8 & 1 & 5 \\ -6 & 21 & 3 \\ 3 & 4 & 1 \end{bmatrix}$$

$$F = \begin{bmatrix} 8 & 1 & 5 \\ 8 & 1 & 1 \\ 3 & 4 & 1 \end{bmatrix}$$ ◀

7. Find an elementary matrix E that satisfies the equation.

(a) $EA = B$ (b) $EB = A$

(c) $EA = C$ (d) $EC = A$

8. Find an elementary matrix E that satisfies the equation.

(a) $EB = D$ (b) $ED = B$

(c) $EB = F$ (d) $EF = B$

▶ In Exercises 9–24, use the inversion algorithm to find the inverse of the given matrix, if the inverse exists. ◀

9. $\begin{bmatrix} 1 & 4 \\ 2 & 7 \end{bmatrix}$ 10. $\begin{bmatrix} -3 & 6 \\ 4 & 5 \end{bmatrix}$ 11. $\begin{bmatrix} -1 & 3 \\ 3 & -2 \end{bmatrix}$

12. $\begin{bmatrix} 6 & -4 \\ -3 & 2 \end{bmatrix}$

13. $\begin{bmatrix} 3 & 4 & -1 \\ 1 & 0 & 3 \\ 2 & 5 & -4 \end{bmatrix}$

14. $\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$

15. $\begin{bmatrix} -1 & 3 & -4 \\ 2 & 4 & 1 \\ -4 & 2 & -9 \end{bmatrix}$

16. $\begin{bmatrix} \frac{1}{5} & \frac{1}{5} & -\frac{2}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{10} \\ \frac{1}{5} & -\frac{4}{5} & \frac{1}{10} \end{bmatrix}$

17. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

18. $\begin{bmatrix} \sqrt{2} & 3\sqrt{2} & 0 \\ -4\sqrt{2} & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

19. $\begin{bmatrix} 2 & 6 & 6 \\ 2 & 7 & 6 \\ 2 & 7 & 7 \end{bmatrix}$

20. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 3 & 5 & 0 \\ 1 & 3 & 5 & 7 \end{bmatrix}$

21. $\begin{bmatrix} 2 & -4 & 0 & 0 \\ 1 & 2 & 12 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & -1 & -4 & -5 \end{bmatrix}$

22. $\begin{bmatrix} -8 & 17 & 2 & \frac{1}{3} \\ 4 & 0 & \frac{2}{5} & -9 \\ 0 & 0 & 0 & 0 \\ -1 & 13 & 4 & 2 \end{bmatrix}$

23. $\begin{bmatrix} -1 & 0 & 1 & 0 \\ 2 & 3 & -2 & 6 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 5 \end{bmatrix}$

24. $\begin{bmatrix} 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & -1 & 3 & 0 \\ 2 & 1 & 5 & -3 \end{bmatrix}$

In Exercises 25–26, find the inverse of each of the following 4×4 matrices, where k_1, k_2, k_3, k_4, and k are all nonzero.

25. (a) $\begin{bmatrix} k_1 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 \\ 0 & 0 & k_3 & 0 \\ 0 & 0 & 0 & k_4 \end{bmatrix}$ (b) $\begin{bmatrix} k & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & k & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

26. (a) $\begin{bmatrix} 0 & 0 & 0 & k_1 \\ 0 & 0 & k_2 & 0 \\ 0 & k_3 & 0 & 0 \\ k_4 & 0 & 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} k & 0 & 0 & 0 \\ 1 & k & 0 & 0 \\ 0 & 1 & k & 0 \\ 0 & 0 & 1 & k \end{bmatrix}$

In Exercises 27–28, find all values of c, if any, for which the given matrix is invertible.

27. $\begin{bmatrix} c & c & c \\ 1 & c & c \\ 1 & 1 & c \end{bmatrix}$

28. $\begin{bmatrix} c & 1 & 0 \\ 1 & c & 1 \\ 0 & 1 & c \end{bmatrix}$

In Exercises 29–32, write the given matrix as a product of elementary matrices.

29. $\begin{bmatrix} -3 & 1 \\ 2 & 2 \end{bmatrix}$

30. $\begin{bmatrix} 1 & 0 \\ -5 & 2 \end{bmatrix}$

31. $\begin{bmatrix} 1 & 0 & -2 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{bmatrix}$

32. $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

In Exercises 33–36, write the *inverse* of the given matrix as a product of elementary matrices.

33. The matrix in Exercise 29.

34. The matrix in Exercise 30.

35. The matrix in Exercise 31.

36. The matrix in Exercise 32.

In Exercises 37–38, show that the given matrices A and B are row equivalent, and find a sequence of elementary row operations that produces B from A.

37. $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 1 \\ 2 & 1 & 9 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 2 & -2 \\ 1 & 1 & 4 \end{bmatrix}$

38. $A = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 1 & 0 \\ 3 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 6 & 9 & 4 \\ -5 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix}$

39. Show that if

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & c \end{bmatrix}$$

is an elementary matrix, then at least one entry in the third row must be zero.

40. Show that

$$A = \begin{bmatrix} 0 & a & 0 & 0 & 0 \\ b & 0 & c & 0 & 0 \\ 0 & d & 0 & e & 0 \\ 0 & 0 & f & 0 & g \\ 0 & 0 & 0 & h & 0 \end{bmatrix}$$

is not invertible for any values of the entries.

41. Prove that if A and B are $m \times n$ matrices, then A and B are row equivalent if and only if A and B have the same reduced row echelon form.

42. Prove that if A is an invertible matrix and B is row equivalent to A, then B is also invertible.

43. Show that if B is obtained from A by performing a sequence of elementary row operations, then there is a second sequence of elementary row operations, which when applied to B recovers A.

True-False Exercises

In parts (a)–(g) determine whether the statement is true or false, and justify your answer.

(a) The product of two elementary matrices of the same size must be an elementary matrix.

(b) Every elementary matrix is invertible.

(c) If A and B are row equivalent, and if B and C are row equivalent, then A and C are row equivalent.

(d) If A is an $n \times n$ matrix that is not invertible, then the linear system $A\mathbf{x} = 0$ has infinitely many solutions.

(e) If A is an $n \times n$ matrix that is not invertible, then the matrix obtained by interchanging two rows of A cannot be invertible.

(f) If A is invertible and a multiple of the first row of A is added to the second row, then the resulting matrix is invertible.

(g) An expression of the invertible matrix A as a product of elementary matrices is unique.

1.6 More on Linear Systems and Invertible Matrics

In this section we will show how the inverse of a matrix can be used to solve a linear system and we will develop some more results about invertible matrices.

Number of Solutions of a Linear System

In Section 1.1 we made the statement (based on Figures 1.1.1 and 1.1.2) that every linear system has either no solutions, has exactly one solution, or has infinitely many solutions. We are now in a position to prove this fundamental result.

THEOREM 1.6.1 *A system of linear equations has zero, one, or infinitely many solutions. There are no other possibilities.*

Proof If $A\mathbf{x} = \mathbf{b}$ is a system of linear equations, exactly one of the following is true: (a) the system has no solutions, (b) the system has exactly one solution, or (c) the system has more than one solution. The proof will be complete if we can show that the system has infinitely many solutions in case (c).

Assume that $A\mathbf{x} = \mathbf{b}$ has more than one solution, and let $\mathbf{x}_0 = \mathbf{x}_1 - \mathbf{x}_2$, where $\mathbf{x}_1$ and $\mathbf{x}_2$ are any two distinct solutions. Because $\mathbf{x}_1$ and $\mathbf{x}_2$ are distinct, the matrix $\mathbf{x}_0$ is nonzero; moreover,

$$A\mathbf{x}_0 = A(\mathbf{x}_1 - \mathbf{x}_2) = A\mathbf{x}_1 - A\mathbf{x}_2 = \mathbf{b} - \mathbf{b} = \mathbf{0}$$

If we now let k be any scalar, then

$$\begin{aligned} A(\mathbf{x}_1 + k\mathbf{x}_0) &= A\mathbf{x}_1 + A(k\mathbf{x}_0) = A\mathbf{x}_1 + k(A\mathbf{x}_0) \\ &= \mathbf{b} + k\mathbf{0} = \mathbf{b} + \mathbf{0} = \mathbf{b} \end{aligned}$$

But this says that $\mathbf{x}_1 + k\mathbf{x}_0$ is a solution of $A\mathbf{x} = \mathbf{b}$. Since $\mathbf{x}_0$ is nonzero and there are infinitely many choices for k, the system $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions. ◀

Solving Linear Systems by Matrix Inversion

Thus far we have studied two *procedures* for solving linear systems—Gauss–Jordan elimination and Gaussian elimination. The following theorem provides an actual *formula* for the solution of a linear system of n equations in n unknowns in the case where the coefficient matrix is invertible.

THEOREM 1.6.2 *If A is an invertible $n \times n$ matrix, then for each $n \times 1$ matrix $\mathbf{b}$, the system of equations $A\mathbf{x} = \mathbf{b}$ has exactly one solution, namely, $\mathbf{x} = A^{-1}\mathbf{b}$.*

▶ EXAMPLE 4 Determining Consistency by Elimination

What conditions must b_1, b_2, and b_3 satisfy in order for the system of equations

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= b_1 \\ 2x_1 + 5x_2 + 3x_3 &= b_2 \\ x_1 \qquad\;\; + 8x_3 &= b_3 \end{aligned}$$

to be consistent?

Solution The augmented matrix is

$$\begin{bmatrix} 1 & 2 & 3 & b_1 \\ 2 & 5 & 3 & b_2 \\ 1 & 0 & 8 & b_3 \end{bmatrix}$$

Reducing this to reduced row echelon form yields (verify)

$$\begin{bmatrix} 1 & 0 & 0 & -40b_1 + 16b_2 + 9b_3 \\ 0 & 1 & 0 & 13b_1 - 5b_2 - 3b_3 \\ 0 & 0 & 1 & 5b_1 - 2b_2 - b_3 \end{bmatrix} \tag{2}$$

In this case there are no restrictions on b_1, b_2, and b_3, so the system has the unique solution

$$x_1 = -40b_1 + 16b_2 + 9b_3, \quad x_2 = 13b_1 - 5b_2 - 3b_3, \quad x_3 = 5b_1 - 2b_2 - b_3 \tag{3}$$

for all values of b_1, b_2, and b_3. ◀

What does the result in Example 4 tell you about the coefficient matrix of the system?

Skills

- Determine whether a linear system of equations has no solutions, exactly one solution, or infinitely many solutions.
- Solve linear systems by inverting its coefficient matrix.
- Solve multiple linear systems with the same coefficient matrix simultaneously.
- Be familiar with the additional conditions of invertibility stated in the Equivalence Theorem.

Exercise Set 1.6

▶ In Exercises **1–8**, solve the system by inverting the coefficient matrix and using Theorem 1.6.2. ◀

1. $x_1 + x_2 = 2$
$5x_1 + 6x_2 = 9$

2. $4x_1 - 3x_2 = -3$
$2x_1 - 5x_2 = 9$

3. $x_1 + 3x_2 + x_3 = 4$
$2x_1 + 2x_2 + x_3 = -1$
$2x_1 + 3x_2 + x_3 = 3$

4. $5x_1 + 3x_2 + 2x_3 = 4$
$3x_1 + 3x_2 + 2x_3 = 2$
$x_2 + x_3 = 5$

5. $x + y + z = 5$
$x + y - 4z = 10$
$-4x + y + z = 0$

6. $-x - 2y - 3z = 0$
$w + x + 4y + 4z = 7$
$w + 3x + 7y + 9z = 4$
$-w - 2x - 4y - 6z = 6$

7. $3x_1 + 5x_2 = b_1$
$x_1 + 2x_2 = b_2$

8. $x_1 + 2x_2 + 3x_3 = b_1$
$2x_1 + 5x_2 + 5x_3 = b_2$
$3x_1 + 5x_2 + 8x_3 = b_3$

▶ In Exercises **9–12**, solve the linear systems together by reducing the appropriate augmented matrix. ◀

9. $x_1 - 5x_2 = b_1$
$3x_1 + 2x_2 = b_2$
(i) $b_1 = 1, \; b_2 = 4$ (ii) $b_1 = -2, \; b_2 = 5$

10. $-x_1 + 4x_2 + x_3 = b_1$
$x_1 + 9x_2 - 2x_3 = b_2$
$6x_1 + 4x_2 - 8x_3 = b_3$
(i) $b_1 = 0, \; b_2 = 1, \; b_3 = 0$
(ii) $b_1 = -3, \; b_2 = 4, \; b_3 = -5$

11. $4x_1 - 7x_2 = b_1$
$x_1 + 2x_2 = b_2$
(i) $b_1 = 0, \; b_2 = 1$ (ii) $b_1 = -4, \; b_2 = 6$
(iii) $b_1 = -1, \; b_2 = 3$ (iv) $b_1 = -5, \; b_2 = 1$

12. $$\begin{aligned} x_1 + 3x_2 + 5x_3 &= b_1 \\ -x_1 - 2x_2 &= b_2 \\ 2x_1 + 5x_2 + 4x_3 &= b_3 \end{aligned}$$

(i) $b_1 = 1,\ b_2 = 0,\ b_3 = -1$
(ii) $b_1 = 0,\ b_2 = 1,\ b_3 = 1$
(iii) $b_1 = -1,\ b_2 = -1,\ b_3 = 0$

In Exercises 13–17, determine conditions on the b_i's, if any, in order to guarantee that the linear system is consistent.

13. $$\begin{aligned} x_1 + 3x_2 &= b_1 \\ -2x_1 + x_2 &= b_2 \end{aligned}$$

14. $$\begin{aligned} 6x_1 - 4x_2 &= b_1 \\ 3x_1 - 2x_2 &= b_2 \end{aligned}$$

15. $$\begin{aligned} x_1 - 2x_2 + 5x_3 &= b_1 \\ 4x_1 - 5x_2 + 8x_3 &= b_2 \\ -3x_1 + 3x_2 - 3x_3 &= b_3 \end{aligned}$$

16. $$\begin{aligned} x_1 - 2x_2 - x_3 &= b_1 \\ -4x_1 + 5x_2 + 2x_3 &= b_2 \\ -4x_1 + 7x_2 + 4x_3 &= b_3 \end{aligned}$$

17. $$\begin{aligned} x_1 - x_2 + 3x_3 + 2x_4 &= b_1 \\ -2x_1 + x_2 + 5x_3 + x_4 &= b_2 \\ -3x_1 + 2x_2 + 2x_3 - x_4 &= b_3 \\ 4x_1 - 3x_2 + x_3 + 3x_4 &= b_4 \end{aligned}$$

18. Consider the matrices

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 2 & 2 & -2 \\ 3 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

(a) Show that the equation $A\mathbf{x} = \mathbf{x}$ can be rewritten as $(A - I)\mathbf{x} = \mathbf{0}$ and use this result to solve $A\mathbf{x} = \mathbf{x}$ for $\mathbf{x}$.

(b) Solve $A\mathbf{x} = 4\mathbf{x}$.

In Exercises 19–20, solve the given matrix equation for X.

19. $$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 0 & 2 & -1 \end{bmatrix} X = \begin{bmatrix} 2 & -1 & 5 & 7 & 8 \\ 4 & 0 & -3 & 0 & 1 \\ 3 & 5 & -7 & 2 & 1 \end{bmatrix}$$

20. $$\begin{bmatrix} -2 & 0 & 1 \\ 0 & -1 & -1 \\ 1 & 1 & -4 \end{bmatrix} X = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 6 & 7 & 8 & 9 \\ 1 & 3 & 7 & 9 \end{bmatrix}$$

21. Let $A\mathbf{x} = \mathbf{0}$ be a homogeneous system of n linear equations in n unknowns that has only the trivial solution. Show that if k is any positive integer, then the system $A^k\mathbf{x} = \mathbf{0}$ also has only the trivial solution.

22. Let $A\mathbf{x} = \mathbf{0}$ be a homogeneous system of n linear equations in n unknowns, and let Q be an invertible $n \times n$ matrix. Show that $A\mathbf{x} = \mathbf{0}$ has just the trivial solution if and only if $(QA)\mathbf{x} = \mathbf{0}$ has just the trivial solution.

23. Let $A\mathbf{x} = \mathbf{b}$ be any consistent system of linear equations, and let $\mathbf{x}_1$ be a fixed solution. Show that every solution to the system can be written in the form $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_0$, where $\mathbf{x}_0$ is a solution to $A\mathbf{x} = \mathbf{0}$. Show also that every matrix of this form is a solution.

24. Use part (*a*) of Theorem 1.6.3 to prove part (*b*).

True-False Exercises

In parts (a)–(g) determine whether the statement is true or false, and justify your answer.

(a) It is impossible for a system of linear equations to have exactly two solutions.

(b) If the linear system $A\mathbf{x} = \mathbf{b}$ has a unique solution, then the linear system $A\mathbf{x} = \mathbf{c}$ also must have a unique solution.

(c) If A and B are $n \times n$ matrices such that $AB = I_n$, then $BA = I_n$.

(d) If A and B are row equivalent matrices, then the linear systems $A\mathbf{x} = \mathbf{0}$ and $B\mathbf{x} = \mathbf{0}$ have the same solution set.

(e) If A is an $n \times n$ matrix and S is an $n \times n$ invertible matrix, then if $\mathbf{x}$ is a solution to the linear system $(S^{-1}AS)\mathbf{x} = \mathbf{b}$, then $S\mathbf{x}$ is a solution to the linear system $A\mathbf{y} = S\mathbf{b}$.

(f) Let A be an $n \times n$ matrix. The linear system $A\mathbf{x} = 4\mathbf{x}$ has a unique solution if and only if $A - 4I$ is an invertible matrix.

(g) Let A and B be $n \times n$ matrices. If A or B (or both) are not invertible, then neither is AB.

1.7 Diagonal, Triangular, and Symmetric Matrices

In this section we will discuss matrices that have various special forms. These matrices arise in a wide variety of applications and will play an important role in our subsequent work.

Diagonal Matrices

A square matrix in which all the entries off the main diagonal are zero is called a ***diagonal matrix***. Here are some examples:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 \\ 0 & -5 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$