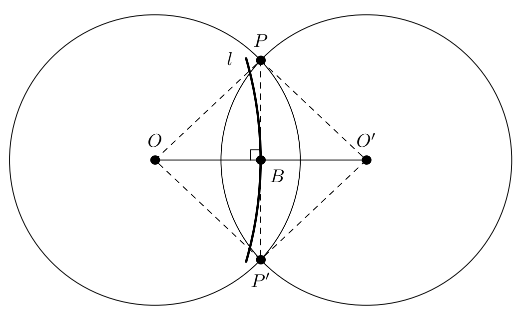
Final (Solutions)

M460 – Geometry

1. [This is the homework from yesterday.] Assume we have two circles, with centers O and O' and same radius, say OR, which intersect in two distinct points, say P' and P. Let B be the midpoint of OO' and l be the line through B perpendicular to $\overleftrightarrow{OO'}$. Assuming that P and P' are in opposite sides of $\overleftrightarrow{OO'}$, show that $P, P' \in l$. **You cannot use any continuity principle!** [I.e., no Circle-Circle, Line-Circle, Segment-Circle, Dedekind's Axiom, etc.] Note that we do *not* know, at least at first, if B, P and P' are colinear [as the picture seems to indicate], so don't use it!

[**Hint:** Melinda was on the right track. Use congruence of triangles to show that $\overleftrightarrow{PP'} \perp \overleftrightarrow{OO'}$ and $B \in \overleftrightarrow{PP'}$. This should help!]



Proof. We have that $\triangle OPB \cong \triangle O'PB$, by SSS, since OP and $O'P$ are both congruent to the radius, and B is the midpoint of OO'. Thus, $\measuredangle PBO \cong \measuredangle PBO'$. Since $O * B * O'$, we get that these angles must be right angles.

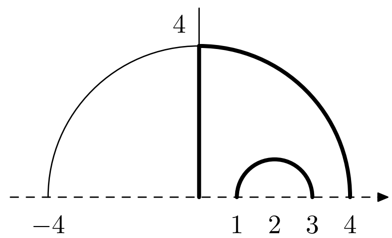
In a similar way, we get that $\measuredangle P'BO$ and $\measuredangle P'BO'$ are also both right angles. Since P and P' are on opposite sides of $\overleftrightarrow{OO'}$ and $\measuredangle PBO$ and $\measuredangle OBP'$ are right angles, we must have [by B-4] that $P * B * P'$ and $\overleftrightarrow{PP'} \perp \overleftrightarrow{OO'}$.

Thus, we have that $B \in l, \overleftrightarrow{PP'}$ and $l, \overleftrightarrow{PP'} \perp \overleftrightarrow{OO'}$. But there is a *unique* line perpendicular to $\overleftrightarrow{OO'}$ passing through B, and thus $l = \overleftrightarrow{PP'}$ and hence $P, P' \in l$. □

2. Show by giving explicit counterexamples [well drawn pictures, preferably explicitly specifying the radii and centers of circles] that the following statements of Euclidean Geometry *do not hold* in the upper half plane (UHP).

(a) *"There can be no line entirely contained in the interior of angle."*

Solution. Take the angle made by the [upper half of the] circle of radius 4 and center at the origin and the y-axis. [So, a $90°$ angle.] In its interior we have the have the whole [upper half of the] circle of center $(2, 0)$ and radius 1.



□

(b) Remember that circles in the UHP are Euclidean circles entirely contained in the upper half plane [but the real center is below the Euclidean center]. *"Given three non-colinear points, there is a circle passing through all of them."*

[**Hint:** There are a couple of different ways to do this. Given three non-colinear points on the UHP, if there is a [non-Euclidean] circle through them, then it is also an Euclidean circle through them *entirely* contained in the UHP. So, if there is no circle through the three non-colinear points, then either there is an Euclidean circle through the points, but it is not contained in the UHP, or there is no [Euclidean] circle at all through the three points.]

Solution. Take the points $(-1, 1)$, $(0, 1)$ and $(1, 1)$. They are not colinear in the UHP, since they are not on a vertical line and there is no Euclidean circle passing to all three [and hence, in particular, no circle with center at the origin], as they are colinear in the Euclidean sense.

But, by the same reason [no Euclidean circle through them], there is no non-Euclidean circle through them! □

3. Consider the distorted model of Problem 35 on pg. 152 [presented in the second project yesterday], where distances on the x-axis are twice as long as they are in the usual $\mathbb{R}^2$ model. [Everything else is the same.]

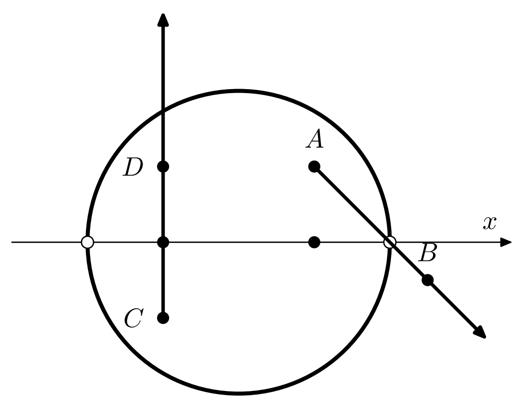
(a) Give an example of a triple (x, y, z) which represents the lengths of three sides of a triangle that exists only if one of its sides is on the x-axis. [**Hint:** *Triangle Inequality* on pg. 171.]

Solution. Consider $(1, 1, 2)$. If the side of length 2 is on the x-axis, then its *usual* length is 1, and hence we have an equilateral triangle in the usual geometry.

If no side is on the x-axis, then we would have a usual triangle with those lengths for the sides. But, by the Triangle Inequality, this is impossible, as $2 \geq 1 + 1$. □

(b) Give examples [with pictures] of rays $\overrightarrow{AB}$ and $\overrightarrow{CD}$ and a circle γ, such that A and C in the interior of γ, $\overrightarrow{AB}$ does not intersect γ and $\overrightarrow{CD}$ intersects γ in exactly two points.

Solution.



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4. Prove that Hilbert's Euclidean Parallel Postulate is equivalent to the transitivity of parallels, i.e., "if $l \parallel m$ and $m \parallel n$, then $l \parallel n$".

[**Hint:** Use Proposition 4.7. In other words, it suffices to show that transitivity of parallels is equivalent to "if $l \parallel m$ and t intersects l, then t also intersects m".]

Proof. Let Statement 1 be "if $l \parallel m$ and $m \parallel n$, then $l \parallel n$", and Statement 2 be "if $l \parallel m$ and t intersects l, then t also intersects m".

Assume, the Statement 1 is true and let $l \parallel m$ and $m \parallel n$. [We need to show that $l \parallel n$.] Suppose that n intersects l [RAA hypothesis]. Then, by Statement 1, since $m \parallel l$, we have that n intersects m, a contradiction. Hence, $n \parallel l$.

Now, assume Statement 2 holds and assume $l \parallel m$ and t intersects l. [We must show that t also intersects m.] Assume that that $t \parallel m$ [RAA hypothesis]. But since $l \parallel m$ and $m \parallel t$, by Statement 2, we should have that $l \parallel t$, which is a contradiction. Therefore, t intersects m. □