1) [15 points] Write the negation of the statement below as a positive statement [i.e., no negation before quantifiers].

$$\forall \epsilon \left[\epsilon > 0 \to \exists N \left(\forall n \left(n \geq N \to \frac{1}{n} < \epsilon \right)\right)\right]$$

[The universe is $\mathbb{R}$ here.]
[You don't *have* to show work here, but skipping steps makes it harder to understand what you've done, so might lead to less partial credit if you got it wrong.]

Solution.

$$\exists \epsilon \left[\epsilon > 0 \land \forall N \left(\exists n \left(n \geq N \land \frac{1}{n} \geq \epsilon \right)\right)\right]$$

□

2) [10 points] Analyze the logical structure of the following statement:

Everyone who has a musician friend knows Pink Floyd.

Careful: Don't make multiple statements into one. For instance, something like

$P(x) = x$ is married to someone who speaks Spanish

is not a simple statement, as it contains two statements: x is married and whoever he is married to speaks Spanish. [Look in the book for examples!]

Solution. Let

$$M(x) = \text{“}x \text{ is a musician”};$$
$$P(x) = \text{“}x \text{ knows Pink Floyd”};$$
$$F(x, y) = \text{“}x \text{ and } y \text{ are friends”}.$$

Then, the statement is:

$$\forall x \left[(\exists y (F(x, y) \land M(y))) \to P(x) \right]$$

□

3) [10 points] Let

$$L(x, y) = x \text{ loves } y.$$

Translate the following statement to [natural sounding] English.

$$\exists x\,[L(\text{Alice}, x) \land \forall y\,([L(\text{Alice}, y) \land y \neq x] \rightarrow y = \text{Alice})]$$

[Try not to lose precision when making it sound natural!]

Solution. "Alice just loves one other person besides herself." □

4) [10 points] Write the truth table for $P \lor Q \rightarrow Q \land \neg R$.

Solution.

P	Q	R	$P \lor Q$	$Q \land \neg R$	$P \lor Q \rightarrow Q \land \neq R$
T	T	T	T	F	F
T	T	F	T	T	T
T	F	T	T	F	F
T	F	F	T	F	T
F	T	T	T	F	F
F	T	F	T	T	T
F	F	T	F	F	T
F	F	F	F	F	T

□

5. 5) [25 points] Let $P \mid Q$ denote "P and Q are not both true".

(a) Write the truth table of $P \mid Q$.

Solution. The truth table is:

P	Q	$P \mid Q$
T	T	F
T	F	T
F	T	T
F	F	T

□

(b) Find a formula [involving P and Q] using only $\wedge$, $\vee$ and $\neg$ operations logically equivalent to $P \mid Q$.

Solution. $\neg(P \wedge Q)$ or $(\neg P) \vee (\neg Q)$. □

(c) Find a formula logically equivalent to $\neg P$ using only $\mid$ [and P]. [*Show* that your formula is indeed equivalent!]

Solution. We have $P \mid P$ works:

$$P \mid P \sim \neg(P \wedge P) \sim \neg P.$$

□

(d) Find a formula for $P \wedge Q$ using only $\mid$ [and P and Q]. [*Show* that your formula is indeed equivalent!]

Solution. We have that $P \mid Q$ is $\neg(P \wedge Q)$ by (b). So by double negatives, we need to negate $P \mid Q$ to get $P \wedge Q$. But, by (c), this is the same as $(P \mid Q) \mid (P \mid Q)$. □

(e) Find a formula for $P \vee Q$ using only $|$ [and P and Q]. [*Show* that your formula is indeed equivalent!]

Solution. By DeMorgan's Law, $P \mid Q \sim (\neq P) \vee (\neg Q)$. So, $(\neg P) \mid (\neg Q) \sim P \vee Q$ [by double negatives]. By part (c), we then have $(P \mid P) \mid (Q \mid Q) \sim P \vee Q$. ☐

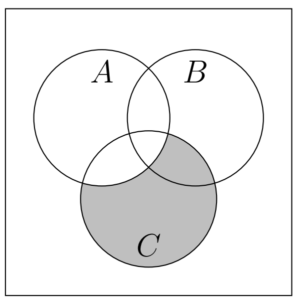
6) [15 points] Let A, B and C be sets and consider:

$$S_1 = (C \setminus A) \cap (A \cup B),$$
$$S_2 = (A \cap C) \cup ((B \cap C) \setminus A).$$

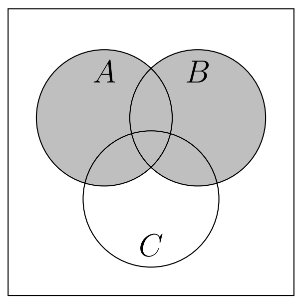
(a) Give the Venn Diagrams for S_1 and S_2. [Label them clearly!]

Solution.

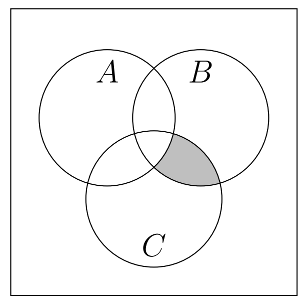
(a) $(C \setminus A) \cap (A \cup B)$:



$C \setminus A$

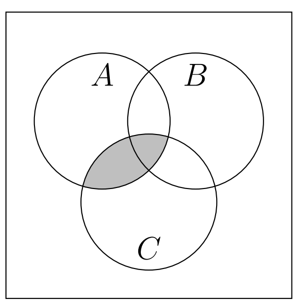


$A \cup B$

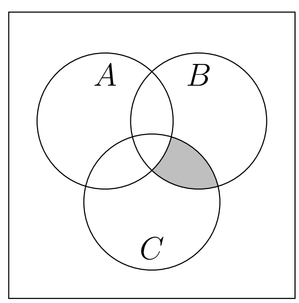


$(C \setminus A) \cap (A \cup B)$

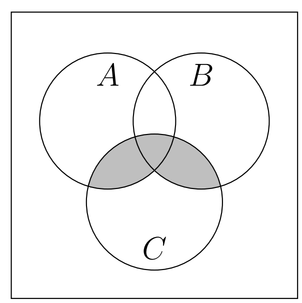
$(A \setminus C) \cup (B \setminus C)$:



$A \cap C$



$(B \cap C) \setminus A$



$(A \cap C) \cup ((B \cap C) \setminus A)$

☐

(b) If the Venn Diagrams above show that $S_1 = S_2$, then show it using logic. If the diagrams are different, give simple *concrete* examples of A, B and C for which the corresponding S_1 and S_2 are different. [*Show work!*]

Proof. We have $S_1 \neq S_2$: let $A = B = C\{1\}$. Then, $C \setminus A = \varnothing$ and so $S_1 = \varnothing$. Now, $A \cap C = \{1\}$ and $(B \cap C) \setminus A = \varnothing$. So, $S_2 = \{1\} \neq \varnothing = S_1$. □

7) [15 points] Let I be a non-empty set of indices and $\{A_i \mid i \in I\}$ and $\{B_i \mid i \in I\}$ be families of sets indexed by I. Prove or disprove:

$$\left(\bigcap_{i\in I} A_i\right) \cap \left(\bigcap_{i\in I} B_i\right) = \bigcap_{i\in I}(A_i \cup B_i).$$

Solution. The statement is false. Let $I = \{1, 2\}$, $A_1 = \{1\}$, $A_2 = \{2\}$, $B_1 = \{2\}$, $B_2 = \{1\}$. Then,

$$\begin{aligned}\left(\bigcap_{i\in I} A_i\right) \cap \left(\bigcap_{i\in I} B_i\right) &= (A_1 \cap A_2) \cap (B_1 \cap B_2)\\ &= \varnothing \cap \varnothing\\ &= \varnothing.\end{aligned}$$

On the other hand:

$$\begin{aligned}\bigcap_{i\in I}(A_i \cup B_i) &= (A_1 \cup B_1) \cap (A_2 \cup B_2)\\ &= \{1, 2\} \cap \{1, 2\}\\ &= \{1, 2\} \neq \varnothing.\end{aligned}$$

□