Midterm 2 Solution

1) [15 points] Let A, B and C be sets. Prove that $(A \cup B) \setminus C \subseteq A \cup (B \setminus C)$.

Proof. Let $x \in (A \cup B) \setminus C$. Then, $x \in A \cup B$ and $x \notin C$. Since $x \in A \cup B$, either $x \in A$ or $x \in B$.
Case 1: Assume $x \in A$. Then, $x \in A \cup (B \setminus C)$.
Case 2: Assume $x \in B$. Since also $x \notin C$, we have that $x \in B \setminus C$ and hence $x \in A \cup (B \setminus C)$.
Thus, for $x \in (A \cup B) \setminus C$, we always have $x \in A(B \setminus C)$. $\square$

2) [15 points] Let $\mathcal{F}$ and $\mathcal{G}$ be non-empty families of sets with $\mathcal{F} \subseteq \mathcal{G}$. Prove that $\bigcap \mathcal{G} \subseteq \bigcap \mathcal{F}$.

Proof. Let $x \in \bigcap \mathcal{G}$. [So, for all $B \in \mathcal{G}$, we have that $x \in B$.] Let $A \in \mathcal{F}$. [We need to show $x \in A$.] Now, since $A \in \mathcal{F}$ and $\mathcal{F} \subseteq \mathcal{G}$, we have that $A \in \mathcal{G}$. Now, as $A \in \mathcal{G}$ and $x \in \bigcap \mathcal{G}$, we have that $x \in A$.
Since $A \in \mathcal{F}$ was arbitrary, we have that $x \in \bigcap \mathcal{F}$. $\square$

3) [15 points] Let R be a relation from A to B and S and T be relations from B to C. Prove that $(S \circ R) \setminus (T \circ R) \subseteq (S \setminus T) \circ R$.

Proof. Let $(a, c) \in (S \circ R) \setminus (T \circ R)$, i.e., $(a, c) \in (S \circ R)$, but $(a, c) \notin (T \circ R)$. So, there is $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. Since $(a, c) \notin (T \circ R)$, we have that $(b, c) \notin T$, and hence $(b, c) \in S \setminus T$. Since $(a, b) \in R$ and $(b, c) \in (S \setminus T)$, we have that $(a, c) \in (S \setminus T) \circ R$. $\square$

4) [15 points] Let R_1 and R_2 be symmetric relations on A. Prove that $R_1 \setminus R_2$ is also symmetric.

Proof. Let $(a, b) \in R_1 \setminus R_2$, i.e., $(a, b) \in R_1$ and $(a, b) \notin R_2$. Since R_1 is symmetric, we have that $(b, a) \in R_1$. Also, since R_2 is symmetric, we have that $(b, a) \notin R_2$, for if $(b, a) \in R_2$, then $(a, b) \in R_2$, which is a contradiction [as $(a, b) \notin R_2$]. Thus, $(b, a) \in R_1 \setminus R_2$. $\square$

5) [20 points] Consider the ordering relation in $\mathbb{R}^2$ defined by $(a, b) \preccurlyeq (c, d)$ [the LaTeX code for this symbol is `\preccurlyeq`] if both $a \leq c$ and $b \leq d$. [You can assume without proving it that this is a partial order in $\mathbb{R}^2$.] Consider the set $B = \{(0, 0)\} \cup \{(1, y) \mid y \in \mathbb{R}\}$. [So, B is the origin together with the vertical line $x = 1$.]

(a) Show that $(0, 0)$ is a minimal element of B.

Proof. Let $(a, b) \in B$ such that $(a, b) \preccurlyeq (0, 0)$. [We need to prove that $(a, b) = (0, 0$.] Then, by definition, we have that $a \leq 0$. Since all elements in B have first coordinate either 0 or 1, we must have $a = 0$. Since $(0, 0)$ is the only element of B with the first coordinate equal to 0, we must have $(a, b) = (0, 0)$. □

(b) Show that B has no other minimal element besides $(0, 0)$.

Proof. Suppose that $(a, b) \in B$ is a minimal element other than $(0, 0)$. Then, we must have $a = 1$ [as all other elements of B have the the first coordinate equal to 1], i.e., $(a, b) = (1, b)$. But then, $(1, b - 1) \preccurlyeq (1, b)$ [by definition of $\preccurlyeq$], and thus $(1, b)$ is not minimal, a contradiction. Thus, $(0, 0)$ is the only minimal element of B. □

(c) Show that B has no smallest element. [In particular, $(0, 0)$ is the only minimal element, but not the smallest element.]

Proof. If B has a smallest element, this would be the only minimal element. Thus, by the above, $(0, 0)$ is the only possibility. But $(1, -1) \in B$ and it is not true that $(0, 0) \preccurlyeq (1, -1)$, and so $(0, 0)$ is not a minimal [and hence, there is no minimal element]. □

6) [20 points] Let $A = \mathbb{R}^2 \setminus \{(0, 0)\}$ [i.e., the Cartesian plane without the origin] and $\mathbb{R}_{>0} = \{x \in \mathbb{R} \mid x > 0\}$ [i.e., the interval $(0, \infty)$]. Define a relation R on A by:

$$R = \{((a, b), (c, d)) \in A \times A \mid \exists x \in \mathbb{R}_{>0} \, (c = ax \wedge d = bx)\}.$$

[I.e., $(a, b) \, R \, (c, d)$ if $(c, d) = (ax, bx)$ for some positive real number x.]

(a) Prove that R is an equivalence relation on A.

Proof. Reflexive: Let $(a,b) \in A$. Since $a = a \cdot 1$, $b = b \cdot 1$ [and $1 \in \mathbb{R}_{>0}$], we have that $(a,b)R(a,b)$ [by definition of R].

Symmetric: Assume that $(a,b)R(c,d)$. Then, [by definition of R] there is $x \in \mathbb{R}_{>0}$ such that $c = ax$ and $d = bx$. Since $x \neq 0$, we have that $a = c \cdot (1/x)$ and $b = d \cdot (1/x)$. Observing that $1/x \in \mathbb{R}_{>0}$ [since $x \in \mathbb{R}_{>0}$], we have that $(c,d)R(a,b)$ [again, by definition of R].

Transitive: Assume that $(a,b)R(c,d)$ and $(c,d)R(e,f)$. Then, there is $x \in \mathbb{R}_{>0}$ such that $c = ax$ and $b = dx$ and $y \in \mathbb{R}_{>0}$ such that $e = cy$ and $f = dy$. Thus, we have that $e = cy = axy$ and $f = dy = bxy$. Since $xy \in \mathbb{R}_{>0}$ [since $x, y \in \mathbb{R}_{>0}$], we have that $(a,b)R(e,f)$.

□

(b) Draw on $A = \mathbb{R}^2 \setminus \{(0,0)\}$ [or describe geometrically] the equivalence class $[(0,1)]_R$.

Solution. We have that $[(0,1)]_R = \{(a,b) \in A \mid a = 0 \cdot x,\ b = 1 \cdot x, \text{ for some } x \in \mathbb{R}_{>0}\} = \{(0,x) \mid x \in \mathbb{R}_{>0}\}$. Hence, geometrically, it is the upper half of the line $x = 0$, not including $(0,0)$. □