Final

This is a take-home exam: You cannot talk to *anyone* (except me) about *anything* on this exam and you can only look at *our* book (Velleman), videos and class notes. *No other reference is allowed*, including the Internet. Failing to follow these instructions will result in a zero for the exam. Moreover, I will report the incident to the university and do all in my power to get the maximal penalty for the infraction.

As you will have a lot of time, I expect the solutions to be well written and clearly explained.

If you need *clarifications on any statement*, please use the "Q&A (Math Related)" forum on Blackboard. *Questions related to content/math of the exam should be submitted by e-mail!* Please use your best judgment on what is appropriate to ask in the forum.

Due date: Your solutions must be uploaded on Blackboard by Wednesday 07/02 by 11:59pm. Please send as a PDF and make sure your scanned/typed exam is clear and legible.

1) [13 points] Let $A \neq \varnothing$ and $f : A \to A$ [so, the codomain is A itself] and assume that for *all* functions $g : A \to A$ we have that $f \circ g = f$. Prove that f is a constant function [i.e., there is $a_0 \in A$ such for all $a \in A$ we have that $f(a) = a_0$].

[**Hint:** What happens if g is constant?]

2) [24 points] Let $A, B \neq \varnothing$ and $f : A \to B$. For $X \subseteq A$, define

$$f(X) = \{f(x) \mid x \in X\}.$$

[**Note:** From this definition we have that $f(\varnothing) = \varnothing$.]

(a) Prove that if $X, Y \subseteq A$, then $f(X \cap Y) \subseteq f(X) \cap f(Y)$.

(b) Give an example for which $f(X \cap Y) \neq f(X) \cap f(Y)$. [**Hint:** There are many examples that work here, but one can make a very simple one where $A = B = \{1, 2\}$. Also, by part (c), note that your example *cannot* be one-to-one!]

(c) Prove that if f is one-to-one, then $f(X \cap Y) = f(X) \cap f(Y)$.

3) [13 points] Let $f : A \to B$ be a one-to-one and onto function, $f^{-1} : B \to A$ be its inverse and $C \subseteq A$, with $C \neq \varnothing$. Prove that $f|C : C \to f(C)$ [with $f|C$ as in Problems 5.1.7 and 5.1.9 and $f(C)$ as in Problem 2 above] is also one-to-one and onto and its inverse is $(f^{-1})|f(C)$.

[**Hint:** This is a *very* simple problem if you can unravel the notation. Just try to not let it overwhelm you!]

4) [16 points] Prove that for all $n \in \mathbb{N}$, we have that $5 \mid (n^5 - n)$.

5) [17 points] Prove that for all $n \in \mathbb{Z}_{\geq 1}$, we have that $5^n \geq 2^n + 3^n$.

6) [17 points] Consider the sequence $a_0, a_1, a_2, \ldots$ given by the recursive formula:

$$\begin{aligned} a_0 &= 1 \\ a_1 &= 1 \\ a_n &= 2a_{n-1} + 3a_{n-2}, \text{ for } n \geq 2. \end{aligned}$$

Prove that for all $n \in \mathbb{N}$, we have that $a_n = (3^n + (-1)^n)/2$.