**1)** [13 points] Let $A \neq \varnothing$ and $f : A \to A$ [so, the codomain is $A$ itself] and assume that for *all* functions $g : A \to A$ we have that $f \circ g = f$. Prove that $f$ is a constant function [i.e., there is $a_0 \in A$ such for all $a \in A$ we have that $f(a) = a_0$].

[**Hint:** What happens if $g$ is constant?]

*Proof.* Since $A \neq \varnothing$, we have that there is $a \in A$. Let $g$ be the function constant equal to $a$, i.e., $g(x) = a$ for all $x \in A$. Then, for all $x \in A$, we have that $(f \circ g)(x) = f(g(x)) = f(a) \in A$. On the other hand, $(f \circ g)(x) = f(x)$ by assumption. So, for every $x \in A$ we have that $f(x) = f(a)$. [So, the $a_0$ is the element $f(a) \in A$.] □

**2)** [24 points] Let $A, B \neq \varnothing$ and $f : A \to B$. For $X \subseteq A$, define

$$f(X) = \{f(x) \mid x \in X\}.$$

[**Note:** From this definition we have that $f(\varnothing) = \varnothing$.]

(a) Prove that if $X, Y \subseteq A$, then $f(X \cap Y) \subseteq f(X) \cap f(Y)$.

*Proof.* Let $b \in f(X \cap Y)$. Then, there is $a \in X \cap Y$, i.e., $a \in X$ and $a \in Y$, such that $b = f(a)$. Since $a \in X$, we have that $b = f(a) \in f(X)$ and since $a \in Y$, we have that $b = f(a) \in f(Y)$. So, $f(X \cap Y) \subseteq f(X) \cap f(Y)$. □

(b) Give an example for which $f(X \cap Y) \neq f(X) \cap f(Y)$. [**Hint:** There are many examples that work here, but one can make a very simple one where $A = B = \{1, 2\}$. Also, by part (c), note that your example *cannot* be one-to-one!]

*Proof.* Let $f : \{1, 2\} \to \{1, 2\}$ given by $f(1) = f(2) = 1$. Let $X = \{1\}$ and $Y = \{2\}$. Then, $f(X \cap Y) = f(\varnothing) = \varnothing$ and $f(X) \cap f(Y) = \{1\} \cap \{1\} = \{1\} \neq \varnothing$. □

(c) Prove that if $f$ is one-to-one, then $f(X \cap Y) = f(X) \cap f(Y)$.

*Proof.* Since from part (a) we already have $f(X \cap Y) \subseteq f(X) \cap f(Y)$, suffices to prove the other inclusion. So, let $b \in f(X) \cap f(Y)$. Hence, there is $b \in f(X)$, i.e., there is $x \in X$ such that $b = f(x)$, and $b \in f(Y)$, i.e., there is $y \in Y$ such that $b = f(y)$. Since $f$ is one-to-one and $f(x) = f(y)$, we must have $x = y$. Since $x \in X$ and $x = y \in Y$, we have $x \in X \cap Y$. Since also $b = f(x)$, we have $b \in f(X \cap Y)$. □

**3)** [13 points] Let $f : A \to B$ be a one-to-one and onto function, $f^{-1} : B \to A$ be its inverse and $C \subseteq A$, with $C \neq \varnothing$. Prove that $f|C : C \to f(C)$ [with $f|C$ as in Problems 5.1.7 and 5.1.9 and $f(C)$ as in Problem 2 above] is also one-to-one and onto and its inverse is $(f^{-1})|f(C)$.

[**Hint:** This is a *very* simple problem if you can unravel the notation. Just try to not let it overwhelm you!]

*Proof.* [One-to-one.] Let $c, c' \in C$ and suppose $f(c) = f(c')$. But, since $f$ is one-to-one, we have that $c = c'$.
[Onto.] Let $b \in f(C)$. Then, there exists $c \in C$ such that $b = f(c)$. So, $f$ in onto.
[Inverse.] Let $b \in f(C)$. Then, there is $c \in C$ such that $b = f(c)$. Since $f$ is invertible, we have then that $c = f^{-1}(b)$. So, $((f^{-1})|f(C))(b) = f^{-1}(b) = c$. Hence, $[(f|C) \circ ((f^{-1})|f(C))](b) = (f|C)(c) = b$, i.e., $[(f|C) \circ ((f^{-1})|f(C))] = i_B$.
Also, if $c \in C$, then $(f|C)(c) = f(c)$ and $((f^{-1})|C)(f(c)) = f^{-1}(f(c)) = c$, and so $[((f^{-1})|f(C)) \circ (f|C)] = i_C$. □

**4)** [16 points] Prove that for all $n \in \mathbb{N}$, we have that $5 \mid (n^5 - n)$.

*Proof.* We prove it by induction on $n$.
[Base case.] For $n = 0$, we have that $5 \mid 0 = 0^5 - 0$.
[Induction step.] Assume now that $5 \mid n^5 - n$ for some $n \geq 0$, i.e., assume that there is $k \in \mathbb{Z}$ such that $n^5 - n = 5k$. [Need to prove that $5 \mid (n+1)^5 - (n+1)$.] We have:

$$\begin{aligned}(n+1)^5 - (n+1) &= n^5 + 5n^4 + 10n^3 + 10n^2 + 5n + 1 - n - 1 \\ &= (n^5 - n) + 5n^4 + 10n^3 + 10n^2 + 5n \\ &= 5k + 5(n^4 + 2n^3 + 2n^2 + n) \\ &= 5(k + n^4 + 2n^3 + 2n^2 + n)\end{aligned}$$

Since $(k + n^4 + 2n^3 + 2n^2 + n) \in \mathbb{Z}$, we have that $5 \mid (n+1)^5 - (n+1)$. □

**5)** [17 points] Prove that for all $n \in \mathbb{Z}_{\geq 1}$, we have that $5^n \geq 2^n + 3^n$.

*Proof.* We prove it by induction on $n$.
[Base case.] We have that $5^1 = 2 + 3 \geq 2^1 + 3^1$.
[Induction step.] Assume now that $5^n \geq 2^n + 3^n$ for some $n \geq 1$. Then,

$$\begin{aligned} 5^{n+1} &= 5 \cdot 5^n \\ &\geq 5 \cdot (2^n + 3^n) && \text{[by IH]} \\ &= 5 \cdot 2^n + 5 \cdot 3^n \\ &\geq 2 \cdot 2^n + 3 \cdot 3^n && [5 \geq 2 \text{ and } 5 \geq 3] \\ &= 2^{n+1} + 3^{n+1}. \end{aligned}$$

□

**6)** [17 points] Consider the sequence $a_0, a_1, a_2, \ldots$ given by the recursive formula:

$$\begin{aligned} a_0 &= 1 \\ a_1 &= 1 \\ a_n &= 2a_{n-1} + 3a_{n-2}, \text{ for } n \geq 2. \end{aligned}$$

Prove that for all $n \in \mathbb{N}$, we have that $a_n = (3^n + (-1)^n)/2$.

*Proof.* We prove it by induction on $n$.
[Base cases.] We have $a_0 = 1 = (3^0 + (-1)^0)/2$. Also, $a_1 = 1 = (3^1 + (-1)^1)/2$.
[Induction step.] Assume now that from some $n \geq 1$ we have that for all $k \in \{0, 1, \ldots, n\}$

that $a_k = (3^{k+} + (-1)^k)/2$. Then,

$$\begin{aligned}
a_{n+1} &= 2a_n + 3 \cdot a_{n-1} \\
&= 2 \cdot \frac{3^n + (-1)^n}{2} + 3 \cdot \frac{3^{n-1} + (-1)^{n-1}}{2} \\
&= \frac{2 \cdot [3^n + (-1)^n] + 3 \cdot [3^{n-1} + (-1)^{n-1}]}{2} \\
&= \frac{[2 \cdot 3^n + 3 \cdot 3^{n-1}] + [2 \cdot (-1)^n + 3 \cdot (-1)^{n-1}]}{2} \\
&= \frac{[2 \cdot 3^n + 3^n] + (-1)^{n-1}[2 \cdot (-1) + 3]}{2} \\
&= \frac{[3 \cdot 3^n] + (-1)^{n-1}}{2} \\
&= \frac{3^{n+1} + (-1)^{n+1}}{2}.
\end{aligned}$$

□