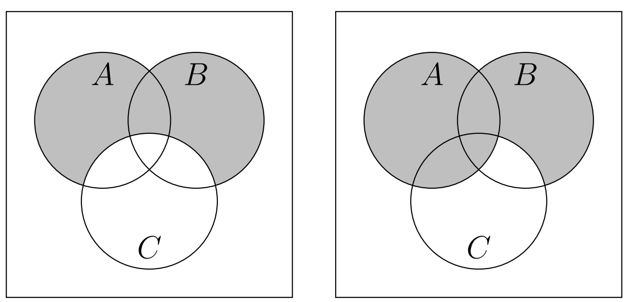
Midterm 1

1) Fill in the [*incomplete*] truth-table below [read the statements carefully!]:

P	Q	R	$(P \wedge Q) \to R$	$Q \vee \neg R$	$[\neg((P \wedge Q) \to R)] \to (Q \vee \neg R)$
T	T	T	T	T	T
T	T	F	F	T	T
T	F	T	T	F	T
T	F	F	T	T	T
F	T	T	T	T	T
F	T	F	T	T	T
F	F	T	T	F	T
F	F	F	T	T	T

2) Prove or disprove: $(A \cup B) \setminus C = A \cup (B \setminus C)$.

Solution. Here are the Venn diagrams:



$(A \cup B) \setminus C$ $\quad$ $A \cup (B \setminus C)$

So, the sets are different. Let then $A = B = C = \{1\}$. Then, $(A \cup B) \setminus C = \{1\} \setminus \{1\} = \varnothing$, but $A \cup (B \setminus C) = \{1\} \setminus \varnothing = \{1\}$. So, the statement is false. $\square$

3) Analyze the logical structure of the following statement: *"There are* exactly *two other people besides Alice who are as smart as she is"*.

You may assume that the universe set is the set of all people, say P, so that you can write, say $\exists x(\ldots)$, instead of $\exists x \in P(\ldots)$, for "there is a person x such that…".

Solution. Let $S(x, y) =$ "x is as smart as y" and let's denote Alice simply by A. Then,

$$\exists x\, [\exists y\, (x \neq A \wedge y \neq A \wedge y \neq x \wedge S(x, A) \wedge S(y, A) \wedge \forall z (S(z, A) \rightarrow (z = x \vee z = y \vee z = A)))]$$

□

4) Rewrite the [nonsensical] statement below as a positive statement [so no negations before quantifiers or parentheses/brackets, but $\notin$ and $\neq$ *are* allowed]. Here the universe is $\mathbb{R}$ [so $\exists x(\ldots)$ means $\exists x \in \mathbb{R}(\ldots)$] and I is the interval $(0, 1)$.

$$\neg\, [\forall x\, [(x \in I \vee x > 10) \leftrightarrow (\exists y (x \cdot y = 1))]]$$

Solution. We have:

$$\begin{aligned}
&\neg\, [\forall x\, [(x \in I \vee x > 10) \leftrightarrow (\exists y (x \cdot y = 1))]] \\
&\quad \sim \exists x \neg\, [(x \in I \vee x > 10) \leftrightarrow (\exists y (x \cdot y = 1))] \\
&\quad \sim \exists x\, [[\neg (x \in I \vee x > 10) \wedge (\exists y (x \cdot y = 1))] \vee [(x \in I \vee x > 10) \wedge \neg\, (\exists y (x \cdot y = 1))]] \\
&\quad \sim \exists x\, [[(x \notin I \wedge x \leq 10) \wedge (\exists y (x \cdot y = 1))] \vee [(x \in I \vee x > 10) \wedge (\forall y (x \cdot y \neq 1))]]\,.
\end{aligned}$$

□

5) Let $\mathcal{F}$ be a family of sets and A be a set. Rewrite the statement

$$\bigcup \mathcal{F} \subseteq \bigcap \mathscr{P}(A),$$

without using $\subseteq$, $\nsubseteq$, $\mathscr{P}$, $\cup$, $\cap$, $\setminus$, $\{$, $\}$ or $\neg$. [You may use $\in$, $\notin$, $=$, $\neq$, $\wedge$, $\vee$, $\rightarrow$, $\forall$ and $\exists$, though.]

Solution.

$$
\begin{aligned}
\bigcup \mathcal{F} \subseteq \bigcap \mathscr{P}(A) &\sim \forall x \in \bigcup \mathcal{F}(x \in \bigcap \mathscr{P}(A)) \\
&\sim \forall x \left[(x \in \bigcup \mathcal{F}) \to (x \in \bigcap \mathscr{P}(A)) \right] \\
&\sim \forall x \left[(\exists X \in \mathcal{F}(x \in X)) \to (\forall Y \in \mathscr{P}(A)(x \in Y)) \right] \\
&\sim \forall x \left[(\exists X \in \mathcal{F}(x \in X)) \to (\forall Y (Y \in \mathscr{P}(A) \to x \in Y)) \right] \\
&\sim \forall x \left[(\exists X \in \mathcal{F}(x \in X)) \to (\forall Y (Y \subseteq A \to x \in Y)) \right] \\
&\sim \forall x \left[(\exists X \in \mathcal{F}(x \in X)) \to (\forall Y ((\forall y \in Y (y \in A)) \to x \in Y)) \right]
\end{aligned}
$$

□

6) Let A and B be sets. Prove that $A \setminus (A \setminus B) = A \cap B$.

Proof. Let $x \in A \setminus (A \setminus B)$. Then, $x \in A$ and $x \notin A \setminus B$. The latter means that either $x \notin A$ or $x \in B$. But, we do have that $x \in A$, so we must have $x \in B$, and hence $x \in A \cap B$. Therefore, we have $A \setminus (A \setminus B) \subseteq A \cap B$.

Now let $x \in A \cap B$. Then, we have that $x \in A$ and $x \in B$. In particular, $x \in A$. Also, since $x \in B$, clearly $x \notin A \setminus B$. So, since $x \in A$ and $x \notin A \setminus B$, we get that $x \in A \setminus (A \setminus B)$. Thus, we've proved that $A \setminus (A \setminus B) \supseteq A \cap B$. Since we had already the other inclusion, we get the equality. □

7) Let $\mathcal{F}$ and $\mathcal{G}$ be non-empty families of sets. Prove that $\bigcup \mathcal{F}$ and $\bigcup \mathcal{G}$ are disjoint iff for every $A \in \mathcal{F}$ and every $B \in \mathcal{G}$ we have that A and B are disjoint.

Proof. [$\to$] Suppose that $\bigcup \mathcal{F} \cap \bigcup \mathcal{G} = \varnothing$ and let $A \in \mathcal{F}$ and $B \in \mathcal{G}$. Suppose that there is $x \in A \cap B$. [So, we should derive a contradiction.] Since $x \in A$ and $A \in \mathcal{F}$, we have [by definition of the union of a family of sets] that $x \in \bigcup \mathcal{F}$. Similarly, since $x \in B$ and $B \in \mathcal{G}$, we have that $x \in \bigcup \mathcal{G}$. Thus, $x \in \bigcup \mathcal{F} \cap \bigcup \mathcal{G}$, a contradiction [as $\bigcup \mathcal{F} \cap \bigcup \mathcal{G} = \varnothing$].

[$\leftarrow$] Now assume that for every $A \in \mathcal{F}$ and every $B \in \mathcal{G}$ we have that A and B are disjoint. Suppose that there is $x \in \bigcup \mathcal{F} \cap \bigcup \mathcal{G}$. Thus, $x \in \bigcup \mathcal{F}$ and $x \in \bigcup \mathcal{G}$. The former says that there is $A \in \mathcal{F}$ such that $x \in A$, while the latter says that there is $B \in \mathcal{G}$ such that $x \in B$. But then $x \in A \cap B = \varnothing$ [by assumption], a contradiction. □

8) Let U be a non-empty set. Prove that for every $A \in \mathscr{P}(U)$, there is a *unique* $B \in \mathscr{P}(U)$ such that for every $C \in \mathscr{P}(U)$ we have $C \setminus A = C \cap B$. [Don't let the $\mathscr{P}(U)$ intimidate you. U here is just "the universe", i.e., all sets in here are contained in this U.]

Proof. [Existence.] Given $A \subseteq U$, let $B = (U \setminus A)$. Then, given $C \subseteq U$, we have that $C \setminus A = C \cap B$ [needs proof!]: let $x \in C \setminus A$. Then, $x \in C$ and $x \notin A$. Since $C \subseteq U$, we have that $x \in U$. Since $x \notin A$, we have that $x \in U \setminus A = B$. Since also $x \in C$, we get $x \in C \cap B$. Conversely, if $x \in C \cup B = C \cup (U \setminus A)$, then $x \in C$ and $x \in U \setminus A$. The last one tells us that [$x \in U$ and] $x \notin A$. Since $x \in C$ also, we have that $x \in C \setminus A$.

[Uniqueness.] Suppose that B' has the same property as $B = U \setminus A$ [for a given A]. [We need to prove that $B' = B$.] Then, taking $C = U$, we have that $U \setminus A = U \cap B' = B'$ [since $B' \subseteq U$.] So, $B = U \setminus A = B'$.

$\square$