1) Suppose that R is a partial order on A, $B_1 \subseteq A$, $B_2 \subseteq A$ and

$$\forall x \in B_1[\exists y \in B_2(xRy)] \qquad \text{and} \qquad \forall x \in B_2[\exists y \in B_1(xRy)].$$

(a) Prove that if $x \in A$ is an upper bound of B_1, then x is also an upper bound of B_2. [The converse is also true, and the proof would very similar, but you don't have to do it.]

Proof. Let x be an upper bound for B_1 and $a \in B_2$. Then, by assumption, there is $b \in B_1$ such that aRb. Now, since x is an upper bound of B_1 and $b \in B_1$, we get bRx. Since R is transitive [as R is a partial order], we get aRx. Hence, x is an upper bound for B_2. □

(b) Prove that if B_1 and B_2 are disjoint, then B_1 has no maximal element. [Again, the same would hold for B_2, but you don't have to do it.]

Proof. Suppose x is a maximal element of B_1. Then, $x \in B_1$ by definition, and so there is $y \in B_2$ such that xRy [by assumption]. But, since $y \in B_2$, there exists $z \in B_1$ such that yRz. But this implies that xRz [by transitivity]. But since x is maximal and $z \in B_1$, we must have zRx. Since R is antisymmetric, we have that $x = z$. But then, since yRz, we have yRx. Since also we had xRy, we get $x = y$, which is a contradiction since $x \in B_1$, $y \in B_2$ and $B_1 \cap B_2 = \varnothing$. □

2) Let $\mathcal{F}$ and $\mathcal{G}$ be partitions of A and let

$$\begin{aligned}\mathcal{H} &= \{Z \in \mathscr{P}(A) \mid Z \neq \varnothing \text{ and } \exists X \in \mathcal{F}[\exists Y \in \mathcal{G}(Z = X \cap Y)]\} \\ &= \{X \cap Y \mid X \cap Y \neq \varnothing,\ X \in \mathcal{F} \text{ and } Y \in \mathcal{G}\}.\end{aligned}$$

Prove that $\mathcal{H}$ is also a partition of A.

Proof. First, note that by definition, no element of $\mathcal{H}$ is empty.
Now, let $a \in A$. Since $\mathcal{F}$ is a partition, there is $X \in \mathcal{F}$ such that $a \in X$. Similarly, since $\mathcal{G}$ is also a partition, we have that there is $Y \in \mathcal{G}$ such that $a \in Y$. Hence, $a \in X \cap Y$ and $X \cap Y \in \mathcal{H}$.

Finally, suppose that $Z, W \in \mathcal{H}$ with $Z \cap W \neq 0$. Since they are in $\mathcal{H}$, there are $X_1, X_2 \in \mathcal{F}$ and $Y_1, Y_2 \in \mathcal{G}$ such that $Z = X_1 \cap Y_1$ and $W = X_2 \cap Y_2$. Since $Z \cap W \neq 0$, let $a \in Z \cap W = X_1 \cap Y_1 \cap X_2 \cap Y_2$. In particular, $a \in X_1 \cap X_2$ and since $\mathcal{F}$ is a partition, we get $X_1 = X_2$. Similarly, since $a \in Y_1 \cap Y_2$ and $\mathcal{G}$ is a partition, we get $Y_1 = Y_2$. Thus, $Z = X_1 \cap Y_1 = X_2 \cap Y_2 = W$. □

3) Let A be a non-empty set, $f : A \to A$. Prove that if f is either a partial order or an equivalence relation, then f is the identity function i_A.

Proof. We need to show that for all $a \in A$, $f(a) = a$, i.e., $(a, a) \in f$. Since either a equivalence relation or a partial order is reflexive, we get that $(a, a) \in f$. □

4) Let $f : A \to B$ be an *invertible* function [i.e., $f^{-1} : B \to B$] and R be an equivalence relation on B. Prove that $S = f^{-1} \circ R \circ f$ is an equivalence relation on A.
[**Hint:** You can use, without proof, the following: $(a, a') \in S$ if there are $b, b' \in B$ such that $(a, b) \in f$, $(b, b') \in R$ and $(b', a') \in f^{-1}$.]

Proof. [Reflexive.] Let $a \in A$. Then, $(a, f(a)) \in f$. Since $f(a) \in B$ and R is reflexive, $(f(a), f(a)) \in R$. Now, since $(a, f(a)) \in f$, we have that $(f(a), a) \in f^{-1}$. Thus, $(a, a) \in S$.

[Symmetric] Suppose that $(a, a') \in S$. Then, there are $b, b' \in B$ such that $(a, b) \in f$, $(b, b') \in R$ and $(b', a') \in f^{-1}$. But, this means that $(b, a) \in f^{-1}$ and $(a', b') \in f$. Also, since R is symmetric, we have that $(b', b) \in R$. So, $(a', a) \in S$.

[Transitive.] Suppose that $(a, a'), (a', a'') \in S$. Then, there are $b, b', b'', b''' \in B$ such that $(a, b), (a', b'') \in f$, $(b, b'), (b'', b''') \in R$ and $(b', a'), (b''', a'') \in f^{-1}$. So, $b = f(a)$, $b'' = f(a')$, bRb', $b''Rb'''$, $b' = f(a')$ and $b''' = f(a'')$. But then, $b' = f(a') = b''$, and so bRb'' [as bRb'']. Since R is transitive [and $b''Rb'''$], we have bRb''', i.e., $(b, b''') \in R$. So, we have $(a, b) \in f$, $(b, b''') \in R$ and $(b''', a'') \in f^{-1}$, and so $(a, a'') \in S$. □

5) Prove that for all integers $n \geq 1$ we have

$$\sum_{i=1}^{n} (2i + 1)3^i = n3^{n+1}.$$

Proof. We prove it by induction on n.

[Base case.] For $n = 1$ we have:

$$(2 \cdot 1 + 1) \cdot 3 = 9 = 1 \cdot 3^{1+1}.$$

[Induction step.] Assume now that for some $n \geq 1$ we have

$$\sum_{i=1}^{n} (2i+1)3^i = n3^{n+1}.$$

Then,

$$\begin{aligned}
\sum_{i=1}^{n+1} (2i+1)3^i &= \left[\sum_{i=1}^{n} (2i+1)3^i\right] + (2(n+1)+1)3^{n+1} \\
&= n3^{n+1} + (2n+3)3^{n+1} \\
&= (n+2n+3)3^{n+1} \\
&= (n+1) \cdot 3 \cdot 3^{n+1} \\
&= (n+1)3^{n+2}.
\end{aligned}$$

□

6) Prove that for all $n \geq 0$ we have

$$\frac{2}{n!} \leq 3^{2-n}.$$

Proof. We prove it by induction on n.

[Base cases.] For $n = 0$ we have $2/0! = 2 \leq 9 = 3^{2-0}$. For $n = 1$, we have $2/1! = 2 \leq 3 = 3^{2-1}$. For $n = 2$, we have $2/2! = 1 \leq 1 = 3^0$.

[Induction step.] Assume that for some $n \geq 2$ we have $2/n! \leq 3^{n-2}$. Then,

$$\begin{aligned}
\frac{2}{(n+1)!} &= \frac{2}{n!} \cdot \frac{1}{n+1} && [(n+1)! = (n+1) \cdot n] \\
&\leq 3^{2-n} \cdot \frac{1}{n+1} && [\text{by IH}] \\
&\leq 3^{2-n} \cdot \frac{1}{3} && [\text{as } n \geq 2] \\
&\leq 3^{2-n-1} = 3^{2-(n+1)}.
\end{aligned}$$

□

7) Consider the sequence $a_0, a_1, a_2, \ldots$ given by the recursive formula:

$$\begin{aligned} a_0 &= 1 \\ a_1 &= 1 \\ a_n &= a_{n-1} + 2a_{n-2}, \text{ for } n \geq 2. \end{aligned}$$

Prove that for all $n \in \mathbb{N}$, we have that $a_n = (2^{n+1} + (-1)^n)/3$.

Proof. We prove it by induction on n.
[Base cases.] We have $a_0 = 1 = (2^1 + (-1)^0)/3$. Also, $a_1 = 1 = (2^2 + (-1)^1)/3$.
[Induction step.] Assume now that from some $n \geq 1$ we have that for all $k \in \{0, 1, \ldots, n\}$ that $a_k = (2^{k+1} + (-1)^k)/3$. Then,

$$\begin{aligned} a_{n+1} &= a_n + 2 \cdot a_{n-1} \\ &= \frac{2^{n+1} + (-1)^n}{3} + 2 \cdot \frac{2^n + (-1)^{n-1}}{3} \\ &= \frac{[2^{n+1} + (-1)^n] + 2 \cdot [2^n + (-1)^{n-1}]}{3} \\ &= \frac{2 \cdot 2^{n+1} + (-1)^n + 2 \cdot (-1)^{n-1}}{3} \\ &= \frac{2^{n+2} + (-1)^{n-1}[-1 + 2]}{3} \\ &= \frac{2^{n+2} + (-1)^{n-1}}{3} \\ &= \frac{2^{n+2} + (-1)^{n+1}}{3}. \end{aligned}$$

□