Exam 2

1) [16 points] Find all the units of $\mathbb{I}_{15}$ and for each unit, find its inverse.

Solution. We have that $[a] \in U(I_{15})$ if and only if $(a, 15) = 1$. So,

$$U(I_{16}) = \{[1], [2], [4], [7], [8], [11], [13], [14]\}.$$

We have:

a	$[a]^{-1}$
1	[1]
2	[8]
4	[4]
7	[13]
8	[2]
11	[11]
13	[7]
14	[14]

□

2) [16 points] Prove that the only subring of $\mathbb{I}_m$ is $\mathbb{I}_m$ itself.

Proof. Let R be a subring of $\mathbb{I}_m$. By definition we have that $[1] \in \mathbb{I}_m$. Then, since R is closed under addition, we have $[1] + [1] = [2] \in R$, and then $[1] + [2] = [3] \in R$, and so on. Hence, we have that $[1], [2], [3], \ldots, [m-1], [m] \in R$. [Note $[m] = [0]$.] So, all elements of $\mathbb{I}_m$ are in R and hence $R = \mathbb{I}_m$ [since $R \subseteq \mathbb{I}_m$]. □

3) [20 points] True or False:

(a) A subring of a field is always a field.

Solution. False. We have that $\mathbb{Z}$ is a subring of $\mathbb{Q}$, but it is not a field. □

(b) A subring of a field is always a domain.

Solution. True. Since a field is always a domain and subrings of domains are domains, we have that subrings of fields are domains. □

4) [16 points] True or False: If F is a field, then there is a domain R with $R \subseteq F$ and $R \neq F$ such that $F = \text{Frac}(R)$.

[**Note:** Remember that $\text{Frac}(R)$ denotes the field of fractions of R. Note also that it is important here that $R \neq F$, for we always have that if F is a field, then $\text{Frac}(F) = F$.]

Solution. False. Since 2 is prime, we have $\mathbb{I}_2$ is a field. By Problem 2, the only subring of $\mathbb{I}_2$ is itself [which can also be easily verified by inspection, as $\mathbb{I}_2 = \{[0], [1]\}$]. So, there is no subring [at all] such that $R \subseteq \mathbb{I}_2$ with $R \neq \mathbb{I}_2$ [and hence none such that $\text{Frac}(R) = \mathbb{I}_2$]. □

5) [16 points] Simplify:

(a) $([1] + [4]x)^3$ in $\mathbb{I}_8[x]$.

Solution.

$$\begin{aligned} ([1] + [4]x)^3 &= [1]^3 + 3 \cdot [1]^2 \cdot [4]x + 3 \cdot [1] \cdot [4]^2 x^2 + [4]^3 x^3 \\ &= [1] + [12]x \\ &= [1] + [4]x. \end{aligned}$$

□

(b) $([1]x^2 + [1]x^3 + [1]x^5)^2$ in $\mathbb{I}_2[x]$.

Solution.

$$\begin{aligned} ([1]x^2 + [1]x^3 + [1]x^5)^2 &= [1]^2x^4 + [1]^2x^6 + [1]^2x^{10} + \\ &\qquad 2 \cdot [1]x^2 \cdot [1]x^3 + 2 \cdot [1]x^2 \cdot [1]x^5 + 2 \cdot [1]x^3 \cdot [1]x^5 \\ &= [1]x^4 + [1]x^6 + [1]x^{10}. \end{aligned}$$

□

(c) $([2]x + [1]x^4)^3$ in $\mathbb{I}_3[x]$

Solution.

$$\begin{aligned} ([2]x + [1]x^4)^3 &= [2]^3x^3 + 3 \cdot [2]^2x^2 + [1]^2x^4 + 3 \cdot [2]x \cdot [1]^2x^8 + [1]^3x^{12} \\ &= [2]x^3 + [1]x^{12}. \end{aligned}$$

□

6) [16 points] Let R be a commutative ring. Prove that $R[x]$ is never a field.

Proof. Assume the $R[x]$ is a field. Then, since R is a subring of $R[x]$, we have that R is a domain [as seen in Problem 3]. Then, for $f, g \in R[x] \setminus \{0\}$, we have that $\deg(f \cdot g) = \deg(f) + \deg(g)$.

Now, since x is a unit [as $x \neq 0$ and $R[x]$ is a field], there is $f \in R[x]$ such that $x \cdot f = 1$. But then,

$$1 + \deg(f) = \deg(x) + \deg(f) = \deg(x \cdot f) = \deg(1) = 0$$

But this implies that $\deg(f) = -1$, which is a contradiction. Thus, $R[x]$ cannot be a field. $\square$