# Exam 3

You must upload the solutions to this exam by 11:59pm on *Sunday* 08/06. [Note it was postponed!] Since this is a take home, I want all your solutions to be neat and well written.

**You can look at class discussions on Cocalc, my videos (now allowed) and *our* book only (*except* for the hints to exercises in the back of the book)!** You *cannot* look at solutions posted by me or *any* other references (including the Internet) without my previous approval. Also, of course, you cannot discuss this with *anyone*!

You can use a computer only to check your answers, as **you need to show work in all questions**.

**1)** [20 points] Use the *Euclidean Algorithm* to find the GCD of $f = x^7 + x^6 + 2x^4 + x^3 + x^2 + x + 2$ and $g = x^6 + 2x^5 + x^4 + x^3 + x^2 + 2x + 1$ in $\mathbb{F}_3[x]$. [Not in $\mathbb{Z}[x]$!]

**2)** [20 points] Let $k$ be a field and $p_1, p_2, p_3, p_4 \in k[x]$ be monic irreducible polynomials in $k[x]$. Suppose that

$$f = a \cdot p_1^2 \cdot p_2^r \cdot p_3^s \qquad \text{and} \qquad g = b \cdot p_1^t \cdot p_2^3 \cdot p_4,$$

where $a, b \in k$, $a, b \neq 0$, and $r$, $s$ and $t$ are non-negative integers. If we *know* that

$$\gcd(f, g) = p_1 \cdot p_2^3 \qquad \text{and} \qquad \operatorname{lcm}(f, g) = p_1^2 \cdot p_2^3 \cdot p_3^5 \cdot p_4,$$

then what are $r$, $s$ and $t$?

**3)** [20 points] Let $k$ be a field and $f$ and $g$ be distinct monic irreducible polynomials in $k[x]$. Prove that the polynomials $f^2 \cdot g^3$ and $f^3 \cdot g^2$ are *never* equal.

[**Hint:** If you are having a hard time figuring out, try to see what this would say in terms of integers instead of polynomials.]

**4)** Examples:

(a) [10 points] Give an example of a domain $R$ such that $R$ is a subring of $\mathbb{F}_2(x)$, but $R$ is *not* a field.

(b) [10 points] Give an example of a field $F$ that contains $\mathbb{C}(x)$ properly [i.e., a field $F$ that contains $\mathbb{C}(x)$ but is different from $\mathbb{C}(x)$ itself].

**Continues on next page!**

**5)** Determine if the polynomials below are irreducible or not in the corresponding polynomial ring. *Justify each answer!*

(a) [4 points] $f = x^7 + 6x^6 - 27x^4 + 120x^3 - 3x - 15$ in $\mathbb{Q}[x]$.

(b) [4 points] $f = x^4 + x + 1 \in \mathbb{F}_5[x]$.

(c) [4 points] $f = \pi^2\, x - \sqrt{137}$ in $\mathbb{R}[x]$.

(d) [4 points] $f = x^6 - 5x^5 - 2x^4 - 4x^2 + x + 1$ in $\mathbb{Q}[x]$.

(e) [4 points] $f = 304x^3 + 123x^2 - 34x + 90001$ in $\mathbb{Q}[x]$.