# Exam 4

You must upload the solutions to this exam by 11:59pm on *Friday* 08/11. Since this is a take home, I want all your solutions to be neat and well written.

**You can look at class discussions on Cocalc, my videos (now allowed) and *our* book only (*except* for the hints to exercises in the back of the book)!** You *cannot* look at solutions posted by me or *any* other references (including the Internet) without my previous approval. Also, of course, you cannot discuss this with *anyone*!

You can use a computer only to check your answers, as **you need to show work in all questions**.

**Note:** In all "True or False" questions, you need to justify your answer. [Usually a proof if True and a counter-example if False.]

**1)** [40 points] Let $\sigma, \tau \in S_9$ be given by

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 5 & 4 & 1 & 9 & 6 & 3 & 2 & 8 \end{pmatrix} \quad \text{and} \quad \tau = (1\ 5)(3\ 2\ 4\ 7)(6\ 8\ 9).$$

(a) Write the complete factorization of $\sigma$ into disjoint cycles.

(b) Write $\tau$ is matrix form.

(c) Compute $\sigma^{-1}$. [Your answer *must* be in *disjoint cycles form*!]

(d) Compute $\sigma\tau$. [Your answer *must* be in *disjoint cycles form*!]

(e) Compute $\sigma\tau\sigma^{-1}$. [Your answer *must* be in *disjoint cycles form*!]

(f) Write $\tau$ as a product of transpositions.

(g) Compute $\text{sign}(\tau)$.

(h) Compute $|\tau|$.

**2)** Decide if True or False [*with justifications!*].

(a) [7 points] The set of real numbers $\mathbb{R}$ is a group with multiplication.

(b) [8 points] Every infinite group has an element of infinite order.

[**Hint:** Every ring is a group with addition. So, we have lots of examples of groups to think of.]

**3)** [15 points] Let $G$ be a group [with *multiplicative* notation], $m$ and $n$ be positive integers such that $\gcd(m,n) = 1$, and $x \in G$ such that $x^m = x^n = e$ [where $e$ is the identity element, i.e., the "1" of the group]. Prove that $x = e$.

[**Hint:** Use the *Extended Euclidean Algorithm* [or what I call *Bezout's Theorem*] for $m$ and $n$. What is then $x^1$? [Think of two ways to find what it is. Of course, they have to be equal to each other, even if the *look* different.] Also, Corollary 2.50 might come handy.]

**4)** [15 points] Let $G = \mathbb{Q}(x,y) \setminus \{0\}$ [i.e., the set of rational functions on $x$ and $y$ and rational coefficients, except for 0] and

$$H = \{ax^m y^n \ : \ a \in \mathbb{Q} \setminus \{0\} \text{ and } m, n \in \mathbb{Z}\}.$$

[Note that $m$ and $n$ can be zero or negative!] Prove that $H$ is a subgroup of $G$. [Of course, $G$ and $H$ are *multiplicative* groups, as they are not groups with respect to addition.]

**5)** [15 points] Let $p$ be a prime and $G$ be a group of order $p^2$. Prove that $G$ has an element of order $p$.

[**Hint:** What are the possible orders of elements in $G$? What elements have order 1? You can also use Problem 2.40 [without solving it].]