Exam 1

1) Fill in the [*incomplete*] truth-table below:

P	Q	R	$P \wedge Q$	$Q \vee \neg R$	$(P \wedge Q) \to (Q \vee \neg R)$
T	T	T	T	T	T
T	T	F	T	T	T
T	F	T	F	F	T
T	F	F	F	T	T
F	T	T	F	T	T
F	T	F	F	T	T
F	F	T	F	F	T
F	F	F	F	T	T

2) Simplify the expression below using the formulas from pgs. 21 and 23 from the textbook:

$$\neg(\neg P \wedge Q) \vee [(P \wedge \neg R) \vee (Q \wedge \neg Q)]$$

[**Hint:** It should simplify to $\neg Q \vee P$.]

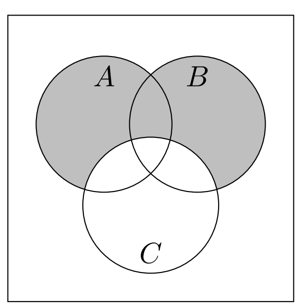
Solution. We have:

$$\begin{aligned} \neg(\neg P \wedge Q) \vee [(P \wedge \neg R) \vee (Q \wedge \neg Q)] &\sim (\neg(\neg P) \vee \neg Q) \vee [(P \wedge \neg R) \vee (Q \wedge \neg Q)] \\ &\sim (P \vee \neg Q) \vee [(P \wedge \neg R) \vee (\text{contr.})] \\ &\sim (P \vee \neg Q) \vee [(P \wedge \neg R)] \\ &\sim (\neg Q \vee P) \vee (P \wedge \neg R) \\ &\sim \neg Q \vee [P \vee (P \wedge \neg R)] \\ &\sim \neg Q \vee P. \end{aligned}$$

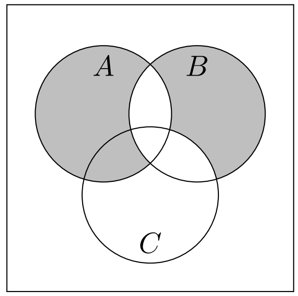
□

3) Show that the sets $(A \cup B) \setminus C$ and $(A \setminus B) \cup (B \setminus (A \cup C))$ are not equal [in general], by giving a *concrete* counterexample.
[**Note:** Drawing the Venn Diagrams is not enough! Although it would give you some partial credit and might help finding the counterexample.]

Solution. We have that $(A \cup B) \setminus C$:



And $(A \setminus B) \cup (B \setminus (A \cup C))$:



So, let $A = B = \{1\}$ and $C = \varnothing$. Then,

$$A \cup B \setminus C = \{1\},$$
$$(A \setminus B) \cup (B \setminus (A \cup C)) = \varnothing \cup (\{1\} \setminus \{1\}) = \varnothing.$$

So, the sets are different. □

4) Express the [nonsensical] statement [with universe of discourse $\mathbb{R}$]

$$\neg [\exists x \ (\forall a \in \mathbb{Z} \ (a \leq x \to [\exists b \in \mathbb{Z} \ (a + b = x)]))] .$$

as a positive statement.

Solution. We have:

$$\begin{aligned}
&\neg [\exists x \ (\forall a \in \mathbb{Z} \ (a \leq x \to [\exists b \in \mathbb{Z} \ (a + b = x)]))] \\
&\quad \sim \forall x \ \neg (\forall a \in \mathbb{Z} \ (a \leq x \to [\exists b \in \mathbb{Z} \ (a + b = x)])) \\
&\quad \sim \forall x \ (\exists a \in \mathbb{Z} \ \neg (a \leq x \to [\exists b \in \mathbb{Z} \ (a + b = x)])) \\
&\quad \sim \forall x \ (\exists a \in \mathbb{Z} \ ((a \leq x) \land \neg [\exists b \in \mathbb{Z} \ (a + b = x)])) \\
&\quad \sim \forall x \ (\exists a \in \mathbb{Z} \ ((a \leq x) \land [\forall b \in \mathbb{Z} \ \neg (a + b = x)])) \\
&\quad \sim \forall x \ (\exists a \in \mathbb{Z} \ ((a \leq x) \land [\forall b \in \mathbb{Z} \ (a + b \neq x)])) .
\end{aligned}$$

□

5) Analyze the following statement: "Every parent has a child who eats only if no one is watching". You can only use the following statements:

$$\begin{aligned}
P(x, y) &= x \text{ is a parent of } y, \\
E(x) &= x \text{ eats}, \\
W(x, y) &= x \text{ is watching } y.
\end{aligned}$$

You can leave implicit the universe of discourse as the set of all people.

Solution.
$$\forall x \ [(\exists y \ P(x, y)) \to (\exists z \ (P(x, z) \land (E(z) \to [\forall w \ (\neg W(w, z))])))] .$$

□

6) Analyze the logical form of the following statements. [You may use $\in$, $\notin$, $=$, $\neq$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, $\forall$ and $\exists$, but not $\subseteq$, $\nsubseteq$, $\mathscr{P}$, $\cap$, $\cup$, $\setminus$, $\{$, $\}$ or $\neg$.]

(a) $x \in \bigcup \mathcal{F} \setminus \bigcap \mathcal{G}$, where $\mathcal{F}$ and $\mathcal{G}$ are families of sets;

Solution.

$$\begin{aligned} x \in \bigcup \mathcal{F} \setminus \bigcap \mathcal{G} &\sim \Big(x \in \bigcup \mathcal{F}\Big) \wedge \neg \Big(x \in \bigcap \mathcal{G}\Big) \\ &\sim (\exists A \in \mathcal{F} \ \ (x \in A)) \wedge \neg (\forall B \in \mathcal{G} \ \ (x \in B)) \\ &\sim (\exists A \in \mathcal{F} \ \ (x \in A)) \wedge (\exists B \in \mathcal{G} \, \neg (x \in B)) \\ &\sim (\exists A \in \mathcal{F} \ \ (x \in A)) \wedge (\exists B \in \mathcal{G} \ \ (x \notin B)) \, . \end{aligned}$$

□

(b) $X \in \bigcup_{i \in I} \mathscr{P}(A_i \cap B_i)$, where A_i and B_i, for $i \in I$, are indexed families.

Solution.

$$\begin{aligned} X \in \bigcup_{i \in I} \mathscr{P}(A_i \cap B_i) &\sim \exists i \in I \ \ [X \in \mathscr{P}(A_i \cap B_i)] \\ &\sim \exists i \in I \ \ [X \subseteq A_i \cap B_i] \\ &\sim \exists i \in I \ \ [\forall x \in X \ \ (x \in A_i \cap B_i)] \\ &\sim \exists i \in I \ \ [\forall x \in X \ \ (x \in A_i \wedge x \in B_i)] \, . \end{aligned}$$

□