Exam 2

You must upload the solutions to this exam (as a PDF file) on Canvas by 11:59pm on Sunday 06/24. Since this is a take home, I want all your solutions to be neat and well written.

You can look at *our* book only! You *cannot* look at our videos, solutions posted by me or *any* other references (including the Internet) without my previous approval. Also, of course, you cannot discuss this with *anyone*!

1) Prove that if $n \in \mathbb{R}$ and $n^2 \notin \mathbb{Z}$, then $n \notin \mathbb{Z}$.

Proof. We prove using the contrapositive. Assume $n \in \mathbb{Z}$. Since products of integers are integers, we have that $n^2 = n \cdot n \in \mathbb{Z}$. □

2) Let $\mathcal{F}$ and $\mathcal{G}$ be non-empty families of sets. Prove that if $\mathcal{F} \subseteq \mathcal{G}$, then $\bigcup \mathcal{F} \subseteq \bigcup \mathcal{G}$.

Proof. Let $x \in \bigcup \mathcal{F}$. Then, there exists $A \in \mathcal{F}$ such that $x \in A$. Since $A \in \mathcal{F}$ and $\mathcal{F} \subseteq \mathcal{G}$, we have that $A \in \mathcal{G}$. Now, since $x \in A$ and $A \in \mathcal{G}$, we have that $x \in \bigcup G$. □

3) Let A_i, for $i \in I$, be an indexed family of sets, with $I \neq \varnothing$. Prove that

$$\bigcap_{i \in I} A_i \in \bigcap_{i \in I} \mathscr{P}(A_i).$$

Proof. Let $i_0 \in I$. Then, $\bigcap_{i \in I} A_i \subseteq A_{i_0}$ [as if $x \in \bigcap_{i \in I} A_i$, then $x \in A_i$ for all $i \in I$, in particular for $i = i_0$], i.e., $\bigcap_{i \in I} A_i \in \mathscr{P}(A_{i_0})$ [by definition of the power set]. Since $i_0 \in I$ was arbitrary, we have that $\bigcap_{i \in I} A_i \in \bigcap_{i_0 \in I} \mathscr{P}(A_{i_0}) = \bigcap_{i \in I} \mathscr{P}(A_i)$ [since i and i_0 are bound variables]. □

4) Let $\mathcal{F}$ and $\mathcal{G}$ be non-empty families of sets. Prove that $(\bigcup \mathcal{F}) \cap (\bigcup \mathcal{G}) = \varnothing$ if and only if for all $A \in \mathcal{F}$ and for all $B \in \mathcal{G}$ we have that $A \cap B = \varnothing$.

Proof. [$\rightarrow$] Assume that $(\bigcup \mathcal{F}) \cap (\bigcup \mathcal{G}) = \varnothing$. [We will do by contradiction.] Assume also that there are $A \in \mathcal{F}$ and $B \in \mathcal{G}$ with $A \cap B \neq \varnothing$. Let then $x \in A \cap B$. Since $A \in \mathcal{F}$ and $x \in A$, we have that $x \in \bigcup \mathcal{F}$. Similarly, since $x \in B$ and $B \in \mathcal{G}$, we have that $x \in \bigcup \mathcal{G}$. But this implies that $x \in (\bigcup \mathcal{F}) \cap (\bigcup \mathcal{G}) = \varnothing$, and so we have a contradiction. Therefore, for all $A \in \mathcal{F}$ and for all $B \in \mathcal{G}$ we have that $A \cap B = \varnothing$.

[$\leftarrow$] Assume now that for all $A \in \mathcal{F}$ and for all $B \in \mathcal{G}$ we have that $A \cap B = \varnothing$. [We will again use contradiction.] Assume also that $(\bigcup \mathcal{F}) \cap (\bigcup \mathcal{G}) \neq \varnothing$, and so let $x \in (\bigcup \mathcal{F}) \cap (\bigcup \mathcal{G})$. Thus, $x \in \bigcup \mathcal{F}$

and $x \in \bigcup \mathcal{G}$. The former means that there is $A \in \mathcal{F}$ such that $x \in A$, while the latter means that there is $B \in \mathcal{G}$ such that $x \in B$. So, $x \in A \cap B = \varnothing$, a contradiction. Therefore, we must have that $(\bigcup \mathcal{F}) \cap (\bigcup \mathcal{G}) = \varnothing$. $\square$

5) Let $\mathcal{F}$ be a non-empty family of sets. Prove that:

$$B \cup \left(\bigcup \mathcal{F}\right) = \bigcup (\mathcal{F} \cup \{B\}).$$

[**Note:** $\mathcal{F} \cup \{B\}$ is a family of sets that has all the sets of $\mathcal{F}$ and also B.]

Proof. [$\subseteq$] Let $x \in B \cup (\bigcup \mathcal{F})$. So, either $x \in B$ or $x \in \bigcup \mathcal{F}$. [We split in cases.]

If $x \in B$, then, as $B \in \mathcal{F} \cup \{B\}$, we have that $x \in \bigcup(\mathcal{F} \cup B)$.

If $x \in \bigcup \mathcal{F}$, then there is $A \in \mathcal{F}$ such that $x \in A$. But, since $A \in \mathcal{F}$, we have that $A \in \mathcal{F} \cup \{B\}$. So, $x \in \bigcup (\mathcal{F} \cup \{B\})$.

[$\supseteq$] Suppose now that $x \in \bigcup (\mathcal{F} \cup \{B\})$. Then, for some $A \in \mathcal{F} \cup \{B\}$, i.e., $A \in \mathcal{F}$ or $A \in \{B\}$, which means $A = B$, we have that $x \in A$. If $A \in \mathcal{F}$, then $x \in \bigcup \mathcal{F}$. If $A = B$, then $x \in B$. So, $x \in B \cup (\bigcup \mathcal{F})$. $\square$

6) Let $\mathcal{F}$ be a non-empty family of sets such that for any family of sets $\mathcal{G}$ such that $\mathcal{G} \subseteq \mathcal{F}$, we have that $\bigcup \mathcal{G} \in \mathcal{F}$. Prove that there exists a unique $A \in \mathcal{F}$ such that for any $B \in \mathcal{F}$, we have that $B \subseteq A$. [So, A contains every set of $\mathcal{F}$.]

Proof. Since $\mathcal{F} \subseteq \mathcal{F}$, by assumption we have that $\bigcup \mathcal{F} \in \mathcal{F}$. So, let $A = \bigcup \mathcal{F}$. Now, if $B \in \mathcal{F}$, then $B \subseteq \bigcup \mathcal{F} = A$ [as if $x \in B$, then since $B \in \mathcal{F}$, we have that $x \in \bigcup \mathcal{F}$]. This proves the existence.

Now suppose we have some $A' \in \mathcal{F}$ such that for all $B \in \mathcal{F}$, we have that $B \subseteq A'$, i.e., A' has the same property as A. [We need to prove that $A' = A$.] Since $A \in \mathcal{F}$, the above means that $A \subseteq A'$. But, since $A' \in \mathcal{F}$, the property of A [proved above] gives that $A' \subseteq A$. Since $A' \subseteq A$ and $A \subseteq A'$, we have that $A = A'$. $\square$