Exam 3

You must upload the solutions to this exam (as a PDF file) on Canvas by 11:59pm on Sunday 07/01. Since this is a take home, I want all your solutions to be neat and well written.

You can look at *our* book only! You *cannot* look at our videos, solutions posted by me or *any* other references (including the Internet) without my previous approval. Also, of course, you cannot discuss this with *anyone*!

1) Let A_i and B_i be indexed families with $I \neq 0$. Prove that

$$\left(\bigcap_{i\in I} A_i\right) \times \left(\bigcap_{i\in I} B_i\right) \subseteq \bigcap_{i\in I}(A_i \times B_i).$$

Proof. Let $(x, y) \in \left(\bigcap_{i\in I} A_i\right) \times \left(\bigcap_{i\in I} B_i\right)$. Then, $x \in \bigcap_{i\in I} A_i$ and $y \in \bigcap_{i\in I} B_i$. So for all $i \in I$ we have that $x \in A_i$ and $y \in B_i$. Hence, for all $i \in I$ we have that $(x, y) \in A_i \cap B_i$. But this means that $(x, y) \in \bigcap_{i\in I}(A_i \times B_i)$. □

2) Let $A = \{1, 2, 3, 4, 5\}$, $B = \{a, b, c, de\}$, and R and S be relations on $A \times B$ and $B \times A$, respectively, given by:

$$R = \{(1, a), (1, d), (2, c), (2, e), (3, b), (3, d), (3, e), (5, a)\},$$
$$S = \{(a, 2), (b, 1), (b, 4), (d, 4), (d, 5), (e, 1), (e, 4)\}.$$

(a) Give $\text{Dom}(R)$.

Solution. $\text{Dom}(R) = \{1, 2, 3, 5\}$. □

(b) Give $\text{Ran}(S)$.

Solution. $\text{Ran}(S) = \{1, 2, 4, 5\}$. □

(c) Give R^{-1}.

Solution.

$$\begin{aligned} R &= \{(a, 1), (d, 1), (c, 2), (e, 2), (b, 3), (d, 3), (e, 3), (a, 5)\} \\ &= \{(a, 1), (a, 5), (b, 3), (c, 2), (d, 1), (d, 3), (e, 2), (e, 3)\} \end{aligned}$$

□

(d) Give $S \circ R$.

Solution.

$$S \circ R = \{(1,2),(1,4),(1,5),(2,1),(2,4),(3,1),(3,4),(3,5),(5,2)\}.$$

□

3) Let R be a non-empty relation on the non-empty set A.

(a) Prove that if R is reflexive, then $R \subseteq R \circ R$.

Proof. Let $(x,y) \in R$. Since R is reflexive, we have that $(y,y) \in R$. Since $(x,y),(y,y) \in R$, we have that $(x,y) \in R \circ R$. □

(b) Prove that if R is transitive, then $R \circ R \subseteq R$.

Proof. Let $(x,z) \in R \circ R$. Then, there is $y \in A$ such that $(x,y) \in R$ and $(y,z) \in R$. Since R is transitive, we must have also that $(x,z) \in R$. □

4) Suppose that R is partial order on A, $B \subseteq A$, and b be the largest element of B. Prove that b is also a maximal element of B and that it is the only maximal element of B.

Proof. [Maximal] Let $x \in B$ such that bRx. Since $x \in B$ and b is the largest element of b, we have that xRb. Since R is a partial order [and hence antisymmetric], we have that $x = b$. Hence, b is maximal.

[Uniqueness] Now assume that $c \in B$ is another maximal element. [Need $c = b$.] Since b is the largest element of B, we have that cRb. But, since $b \in B$ and c is maximal, we have that $b = c$. □

5) Let $A = \{(x,y) \in \mathbb{Z} \times \mathbb{Z} \mid y \neq 0\}$ [i.e., $A = \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$], and define for $(a,b),(c,d) \in A$ the relation R by $(a,b)R(c,d)$ if $ad = bc$.

(a) Prove that R is an equivalence relation. [**Note:** transitivity is a bit tricky, and requires some algebraic manipulations.]

Proof. [Reflexive] We have that $(a,b)R(a,b)$, since $ab = ba$.

[Symmetric] Suppose that $(a,b)R(c,d)$, i.e., $ad = bc$. But then, $da = cb$, and so $(c,d)R(a,b)$.

[Transitive] Suppose that $(a,b)R(c,d)$ and $(c,d)R(e,f)$. Then, $ad = bc$ and $cf = de$. Multiplying the first equation by f and the second by b, we get $adf = bcf$ and $bcf = bde$. So, $adf = bde$. Since $d \neq 0$, we have $af = be$, and so $(a,b)R(e,f)$. □

(b) For any $b \in \mathbb{Z} \setminus \{0\}$, let $S_b = \{(k,kb) \mid k \in \mathbb{Z} \setminus \{0\}\}$. Prove that $[(1,b)]_R = S_b$.

Proof. [$\subseteq$] Let $(x, y) \in [(1, b)]_R$. Then, $(x, y)R(1, b)$ and hence $xb = y$. Note that since $y \neq 0$, we have that $x \neq 0$. Also, $(x, y) = (x, xb)$, and since $x \neq 0$, we have that $(x, y) \in S_b$.

[$\supseteq$] Let $(x, y) \in S_b$. So, $(x, y) = (k, kb)$ for some $k \in \mathbb{Z} \setminus \{0\}$. Then, $kb = 1 \cdot kb$ and hence $(k, kb)R(1, b)$, and so $(k, kb) = (x, y) \in [(1, b)]_R$. □

6) Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be partitions of A_1 and A_2 respectively, with $A_1 \cap A_2 = \varnothing$. Prove that $\mathcal{F}_1 \cup \mathcal{F}_2$ is a partition of $A_1 \cup A_2$.

Proof. [$\mathcal{F}_1 \cup \mathcal{F}_2 \subseteq \mathscr{P}(A_1 \cup A_2)$] Let $X \in \mathcal{F}_1 \cup \mathcal{F}_2$. Then, either $X \in \mathcal{F}_1$ or $X \in \mathcal{F}_2$. If the former, then $X \subseteq A_1 \subseteq A_1 \cup A_2$, as $\mathcal{F}_1 \subseteq \mathscr{P}(A_1)$. If the latter, then $X \subseteq A_2 \subseteq A_1 \cup A_2$, as $\mathcal{F}_1 \subseteq \mathscr{P}(A_1)$. In either case, we have that $X \in \mathscr{P}(A_1 \cup A_2)$.

[No empty set.] If $\varnothing \in \mathcal{F}_1 \cup \mathcal{F}_2$, then either $\varnothing \in \mathcal{F}_1$ or $\varnothing \in \mathcal{F}_2$. But either is impossible, as $\mathcal{F}_1$ and $\mathcal{F}_2$ are partitions. Thus $\varnothing \notin \mathcal{F}_1 \cup \mathcal{F}_2$.

[Disjoint.] Suppose $X, Y \in \mathcal{F}_1 \cup \mathcal{F}_2$ with $X \cap Y \neq \varnothing$. If both X and Y are either in $\mathcal{F}_1$ or in $\mathcal{F}_2$, then $X = Y$, since $\mathcal{F}_1$ and $\mathcal{F}_2$ are partitions. So, assume that $X \in \mathcal{F}_1$ and $Y \in \mathcal{F}_2$. [The case where $X \in \mathcal{F}_2$ and $Y \in \mathcal{F}_1$ is analogous.] But then $X \subseteq A_1$ and $Y \subseteq A_2$. But if $X \cap Y \neq \varnothing$, there is $x \in X \cap Y$, which means $x \in X \subseteq A_1$ and $x \in Y \subseteq A_2$. But this is a contradiction, as $A_1 \cap A_2 = \varnothing$. So, $X \cap Y = \varnothing$.

[Covers $A_1 \cup A_2$]. Let $x \in A_1 \cup A_2$. Then, either $x \in A_1$ or $x \in A_2$. If the former, since $\mathcal{F}_1$ is a partition of A_1, there is $X \in \mathcal{F}_1 \subseteq \mathcal{F}_1 \cup \mathcal{F}_2$ such that $x \in X$. If the latter, since $\mathcal{F}_2$ is a partition of A_2, there is $X \in \mathcal{F}_2 \subseteq \mathcal{F}_1 \cup \mathcal{F}_2$ such that $x \in X$. So, in either case there is $X \in \mathcal{F}_1 \cup \mathcal{F}_2$ such that $x \in X$. □