Exam 2

You must upload the solutions to this exam by 11:59pm on *Sunday* 07/28. Since this is a take home, I want all your solutions to be neat and well written.

You can look at class discussions on Cocalc and *our* book only (*except* for the hints to exercises in the back of the book)! You *cannot* look at our videos, solutions posted by me or *any* other references (including the Internet) without my previous approval. Also, of course, you cannot discuss this with *anyone*!

You can use a computer only to check your answers, as **you need to show work in all questions**.

1) [15 points] Find all the units of $\mathbb{I}_{14}$ and for each unit, find its inverse.

Solution.

unit	inverse
[1]	[1]
[3]	[5]
[5]	[3]
[9]	[11]
[11]	[9]
[13]	[13]

□

2) [20 points] For all examples below, check *all* the boxes that apply [no need to justify]:

(a) $\mathbb{N} = \{0, 1, 2, \dots\}$: ☐ non-commutative ring, ☐ commutative ring, ☐ domain, ☐ field.

(b) $\mathbb{R}$: ☐ non-commutative ring, ☑ commutative ring, ☑ domain, ☑ field.

(c) $\mathbb{I}_5[x]$: ☐ non-commutative ring, ☑ commutative ring, ☑ domain, ☐ field.

(d) $\mathbb{I}_6[x]$: ☐ non-commutative ring, ☑ commutative ring, ☐ domain, ☐ field.

(e) $M_2(\mathbb{Q})$ [i.e., 2×2 matrices with entries in $\mathbb{Q}$]: ☑ non-commutative ring, ☐ commutative ring, ☐ domain, ☐ field.

3) [15 points] Give the prime field of the following fields [no need to justify]:

(a) $\mathbb{Q}$

Solution. $\mathbb{Q}$ itself. $\square$

(b) $\mathbb{R}(x)$

Solution. $\mathbb{Q}$. $\square$

(c) $\mathbb{F}_p(x, y)$ [Note that $\mathbb{F}_p(x, y)$ is the field of rational functions in two variables. You can see if as the field of fractions of $\mathbb{F}_p(x)[y]$, i.e., $\mathbb{F}_p(x)(y)$.]

Solution. $\mathbb{F}_p$. $\square$

4) Let $R = \mathbb{Z}[\sqrt{3}] = \{a + b\sqrt{3} \ : \ a, b \in \mathbb{Z}\}$.

(a) [10 points] Is R a commutative ring? [Justify!]

Solution. First note $R \subseteq \mathbb{R}$, so it suffices to show it is a subring of $\mathbb{R}$.

Note that $1 = 1 + 0 \cdot \sqrt{3} \in F$.

Now, if $a + b\sqrt{3}, c + d\sqrt{3} \in R$ [and so, $a, b, c, d \in \mathbb{Z}$], then $(a + b\sqrt{3}) - (c + d\sqrt{3}) = (a - c) + (b - d)\sqrt{3} \in R$ as $a - c, b - d \in \mathbb{Z}$ [since $\mathbb{Z}$ is closed under differences].

Also, $(a + b\sqrt{3}) \cdot (c + d\sqrt{3}) = (ac + 3bd) + (ad + bc)\sqrt{3} \in R$, since $(ac - bd), (ad + bc) \in \mathbb{Z}$, as $\mathbb{Q}$ is closed under addition and multiplication.

Hence, R is a subring of $\mathbb{R}$. $\square$

(b) [5 points] Is R an integral domain? [Justify!]

Solution. Yes, since $\mathbb{R}$ is a domain and R is a subring of $\mathbb{R}$, we have that R is a domain. $\square$

(c) [5 points] Is R a field? [Justify!]

Solution. We have that 2 has no inverse in R, since if $a + b\sqrt{3} = \dfrac{1}{2}$, then $2a + 2b\sqrt{3} = 1$, and so $\sqrt{3} = \dfrac{1 - 2a}{2b} \in \mathbb{Q}$ [since $a, b \in \mathbb{Z}$], if $b \neq 0$, which is false. So, we must have that $b = 0$, and hence $a = \dfrac{1}{2}$. But this is also impossible since $a \in \mathbb{Z}$. $\square$

5) Let F be the field of fractions of the Gaussian integers $\mathbb{Z}[i] = \{a + bi : a, b \in \mathbb{Z}\}$. [Remember that i is the complex number with $i^2 = -1$ and that $\mathbb{Z}[i]$ is a domain.]

(a) [5 points] Is $\dfrac{2 - 3i}{3 + 2i} = \dfrac{-1}{i}$ in F? [Show your computations.]

Solution. We have that $(2-3i)\cdot i = 3+2i \neq -(3+2i) = (-1)\cdot(3+2i)$. So they are different. $\square$

(b) [10 points] Let $\alpha = \dfrac{1}{2+i}$ and $\beta = \dfrac{1+i}{2-i}$. Compute $\alpha+\beta$ and $\alpha\cdot\beta$ [in F]. **Your answers should be in the form** $\dfrac{x}{y}$ **with** $x, y \in \mathbb{Z}[i]$**!** [Show work!]

Solution. We have:

$$\alpha+\beta = \frac{1}{2+i} + \frac{1+i}{2-i} = \frac{(2-i)+(2+i)(1+i)}{(2+i)(2-i)} = \frac{(2-i)+(1+3i)}{5} = \frac{3+2i}{5},$$

and

$$\alpha\cdot\beta = \frac{1}{2+i}\cdot\frac{1+i}{2-i} = \frac{1\cdot(1+i)}{(2+i)(2-i)} = \frac{1+i}{5}.$$

$\square$

6) [15 points] Prove that every field is an integral domain.

Proof. Suppose that F is a field, $a \neq 0$, and $ax = ay$. [We need to show that $x = y$.] Since $a \neq 0$ and F is a field, we have that a is a unit, and so there is $a^{-1} \in F$ such that $a \cdot a^{-1} = 1$. Then, $a^{-1}(ax) = a^{-1}(ay)$, so $(a^{-1}a)x = (a^{-1}a)y$. Which implies $1\cdot x = 1 \cdot y$, and thus $x = y$. $\square$