Algebra II

Exponents

Project GRAD SI 2022

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Simple Powers

Let's start with a simple question:

Example

What is 2^4?

Indeed, $2^4 = 16$. How did you compute it?

We compute

$$\underbrace{2 \cdot 2 \cdot 2 \cdot 2}_{\text{4 factors}} = 16.$$

Simple Powers

Just to make sure we remember, let's do one more:

Example

What is 3^5?

It is

$$\underbrace{3 \cdot 3 \cdot 3 \cdot 3 \cdot 3}_{5 \text{ factors}} = 243.$$

General Powers

More generally, if a is a real number and n is a positive integer, then

$$a^n = \underbrace{a \cdot a \cdot a \cdots a}_{n \text{ factors}}.$$

Note that $a^1 = a$.

We call a the base and n the exponent of a^n.

What is $2^3 \cdot 2^2$?

Example

What is $2^3 \cdot 2^2$?

Well, this is not a tricky question:

$$2^3 \cdot 2^2 = 8 \cdot 4 = 32.$$

But let's do it in a different way:

$$2^{3} \cdot 2^{2} = \underbrace{2 \cdot 2 \cdot 2}_{3 \text{ factors}} \cdot \underbrace{2 \cdot 2}_{2 \text{ factors}} = \underbrace{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}_{3 + 2 = 5 \text{ factors}} = 2^{5} = 32.$$

$a^m \cdot a^n$

More generally, we have:

$$a^{m} \cdot a^{n} = \underbrace{a \cdot a \cdots a}_{m \text{ factors}} \cdot \underbrace{a \cdot a \cdots a}_{n \text{ factors}} = \underbrace{a \cdot a \cdots a}_{m+n \text{ factors}} = a^{m+n}.$$

Therefore we have:

Theorem (First Basic Property of Exponents)

If a is a real number and m and n are positive integers, then

$$a^m \cdot a^n = a^{m+n}.$$

Note there is no real need to memorize. It's just counting!

What is $(2^3)^2$?

Example

What is $(2^3)^2$?

Again, not a tricky question:

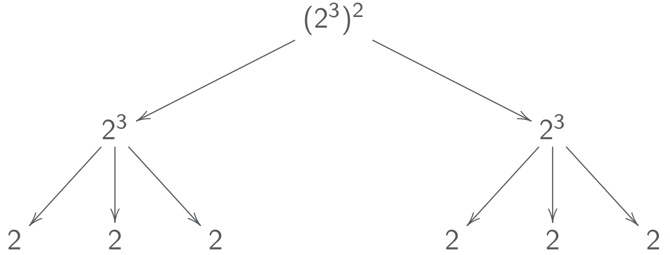
$$(2^3)^2 = 8^2 = 64.$$

But let's again do it in a different way:

$$(2^{3})^{2} = \underbrace{2^{3} \cdot 2^{3}}_{2 \text{ factors}} = \underbrace{2 \cdot 2 \cdot 2}_{3 \text{ factors}} \cdot \underbrace{2 \cdot 2 \cdot 2}_{3 \text{ factors}} = \underbrace{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}_{3 \cdot 2 = 6 \text{ factors}} = 2^{6} = 64.$$

Visualizing

We can see the counting:



[2 factors of 2^3]

[6 factors of 2]

$(a^m)^n$

More generally, we have:

$$(a^m)^n = \underbrace{a^m \cdot a^m \cdots a^m}_{n \text{ factors}} = \underbrace{a \cdot a \cdots a}_{m \cdot n \text{ factors}} = a^{m \cdot n}.$$

Therefore we have:

Theorem (Second Basic Property of Exponents)

If a is a real number and m and n are positive integers, then

$$(a^m)^n = a^{m \cdot n}.$$

Note, again, there is no real need to memorize. It's just counting!

Basic Properties of Exponents

So, we have:

Theorem (Basic Properties of Exponents)

If a is a real number and m and n are positive integers, then:

1. $a^m \cdot a^n = a^{m+n}$;
2. $(a^m)^n = a^{m \cdot n}$.

These are important and you need to remember them!

What is 2^0?

So, we know what a^n means if n is a positive integer (like, 1, 2, 3, 4, and so on). But what is, say, 2^0? (Does anyone remember?)

Note that in principle the question does not make sense: we cannot have 0 factors of 2. But you might have seen this before and in fact is quite useful to give meaning to 2^0.

Since we are the ones choosing what value to give to 2^0, in principle it could have any value. We could say, for instance, that $2^0 = 0$. Or $2^0 = 2$. Or $2^0 = \pi$. Our imagination is the limit!

What is 2^0?

But there is a catch! Depending on hour choice, the Basic Properties of Exponents might stop working if we choose either m or n as 0. So, what we do it to make a choice that make sure that the Basic Properties of Exponents will continue working!

How can we do that? Here is an idea. *If the Basic Properties of Exponents work*, then what do we get when we multiply $2 \cdot 2^0$?

$$2 \cdot 2^0 = 2^1 \cdot 2^0 = 2^{1+0} = 2^1 = 2.$$

In other workds

$$2 \cdot 2^0 = 2.$$

Does this help at all?

What is 2^0?

Well, since we don't know what 2^0 is (but we want to find out), let's call it x. Then

$$2 \cdot 2^0 = 2$$

becomes

$$2x = 2.$$

This equation has exactly one solution, namely $x = 1$. Hence,

$$2^0 = 1.$$

Therefore, the *only* choice for 2^0 that make the Basic Properties of Exponents still work is $2^0 = 1$!

How about 5^0?

How about 5^0? What should it be equal to (if we want the Basic Properties of Exponents to keep working)? Was there anything special about 2? No! We can just replace all 2's by 5's in our computation:

$$5 \cdot 5^0 = 5^1 \cdot 5^0 = 5^{1+0} = 5^1 = 5.$$

So, $5 \cdot 5^0 = 5$ (or, setting $5^0 = x$, we have $5x = 5$). So, we also have

$$5^0 = 1.$$

How about a^0?

How about a^0, where a is any real number now? Do we get $a^0 = 1$, no matter what a is?
It seems like it, as we can do the same:

$$a \cdot a^0 = a^1 \cdot a^0 = a^{1+0} = a^1 = a.$$

So, letting $a^0 = x$, we have

$$ax = a.$$

So, what is x? We have $x = 1$, so $a^0 = 1$. Except, there is a small problem here. Does anyone see it?

Solving $ax = a$

What are we really doing when we "cancel a", to get $x = 1$ from $ax = a$? (What is the algebra step we take?)
We are dividing both sides by a. What is the thing we have to be careful when dividing (by an unknown number)? We cannot divide by 0!
Dividing by 0 would bring all sorts of problems! For instance, we'd have

$$2 \cdot 0 = 0 \qquad \Longrightarrow \qquad 2 = 1.$$

So, we have a problem when $a = 0$. If we repeat the steps, we get:

$$0 \cdot 0^0 = 0^1 \cdot 0^0 = 0^{1+0} = 0^1 = 0.$$

But, letting $x = 0^0$, this gives $0 \cdot x = 0$. What are the possible solutions for this equation? Any value of x works! So there is not a single possible choice. (Note that we cannot have $0/0$!)

What is 0^0?

So, what is 0^0 then? Well, it depends on whom you ask.

Some like to define $0^0 = 1$. This is convenient and is consistent with every other real number. (This can cause some confusion in Calculus, though!)

Others, myself included, prefer to say that 0^0 is undefined. This makes sense since it comes from a division by 0, namely $\frac{0}{0}$ (which is undefined).

So, we will officially say in this course that 0^0 is undefined, but most likely 0^0 will not appear in this course at all.

Exponent 0 and Summary

So, we say

$$a^0 = 1 \text{ for all } a \neq 0.$$

Summarizing what we have so far:

1. If n is a positive integer and a is a real number, then $a^n = \underbrace{a \cdot a \cdots a}_{n \text{ factors}}$.
2. If $a \neq 0$, then $a^0 = 1$. (0^0 is undefined.)
3. If $a \neq 0$ and m and n are non-negative integers (includes 0), then
   - $a^m \cdot a^n = a^{m+n}$;
   - $(a^m)^n = a^{m \cdot n}$.

   (The Basic Properties of Exponents also work for exponent 0!)

Negative Exponents

Since we now have given meaning to a^0, the next natural question is to ask about negative exponents.

Let's start with a concrete example: what should 2^{-1} be? Again, this has no natural meaning (as we can't have "negative one factors of 2"), so we get to decide its value. But just like before we want to make a "good" choice:

We want the Basic Properties of Exponents to work with negative exponents.

What is 2^{-1}?

To find out what 2^{-1} is, we will copy the same idea we used for 2^0: we will start with 2^{-1}, and using the Basic Properties of Exponents, we will try to get rid of the -1 exponent to arrive at a familiar expression.

What can we multiply 2^{-1} by to get rid of the negative exponent? We can multiply by $2 = 2^1$! If the Basic Properties of Exponents work, we have

$$2 \cdot 2^{-1} = 2^1 \cdot 2^{-1} = 2^{1+(-1)} = 2^0 = 1.$$

This means that $2 \cdot 2^{-1} = 1$. If we call $x = 2^{-1}$, this means $2x = 1$. So, what is x? The only x that works is $x = \frac{1}{2}$. So, 2^{-1} must be $\frac{1}{2}$ (if we want the Basic Properties of Exponents to work)!

How about 3^{-1}?

How about 3^{-1}? What should it be? We can copy the same idea! Replacing 2's by 3's in our previous computation we get

$$3 \cdot 3^{-1} = 3^1 \cdot 3^{-1} = 3^{1+(-1)} = 3^0 = 1.$$

This means that $3 \cdot 3^{-1} = 1$. If we call $x = 3^{-1}$, this means $3x = 1$. So, what is x? The only x that works is $x = \frac{1}{3}$. So, 3^{-1} must be $\frac{1}{3}$ (if we want the Basic Properties of Exponents to work)!

How about a^{-1}?

How about the general case of a^{-1} for any number a? What should it be? We can again try to copy the same idea, replacing 2's by a's. We get

$$a \cdot a^{-1} = a^1 \cdot a^{-1} = a^{1+(-1)} = a^0 = 1.$$

This means that $a \cdot a^{-1} = 1$. If we call $x = a^{-1}$, this means $ax = 1$. So, what is x? The only x that works is $x = \frac{1}{a}$. So, a^{-1} must be $\frac{1}{a}$ (if we want the Basic Properties of Exponents to work)!

But there is a problem! What if $a = 0$?

The case of 0^{-1}

There are two problems when $a = 0$:

1. We can't say $a^0 = 1$ if $a = 0$.
2. The faction $\frac{1}{a}$ makes no sense when $a = 0$.

Unlike the case of 0^0 where some left it undefined (like us), while some say $0^0 = 1$, here there is no ambiguity: 0^{-1} is undefined, just like $\frac{1}{0}$!

So, we have:

$$a^{-1} = \frac{1}{a}, \quad \textbf{if } a \neq 0.$$

What is 2^{-2}?

So, now we know that $2^{-1} = 1/2$. Then, the next natural question is: what is 2^{-2}?
This is a bit simpler, if we want the Basic Properties of Exponents to work: we have

$$2^{-2} = 2^{(-1)\cdot 2} = \left(2^{-1}\right)^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{2^2}.$$

Remember:

$$\frac{a}{b}\cdot\frac{c}{d} = \frac{a\cdot c}{b\cdot d}.$$

So, $2^{-2} = \frac{1}{2^2} = \frac{1}{4}$.

What is a^{-n}?

So, in general, we can use the same idea to find a^{-n}, where $a \neq 0$ and n is a positive integer:

$$a^{-n} = a^{(-1)\cdot n} = \left(a^{-1}\right)^n = \left(\frac{1}{a}\right)^n = \underbrace{\frac{1}{a}\cdot\frac{1}{a}\cdots\frac{1}{a}}_{n \text{ factors}} = \frac{1}{a^n}.$$

So,

$$a^{-n} = \frac{1}{a^n} \quad \text{if } a \neq 0.$$

Fractions

Note that we have:

$$\left(\frac{a}{b}\right)^n = \underbrace{\frac{a}{b} \cdot \frac{a}{b} \cdots \frac{a}{b}}_{n \text{ factors}} = \frac{a^n}{b^n}.$$

Also, remember that

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}.$$

So,

$$\left(\frac{a}{b}\right)^{-1} = \frac{1}{\frac{a}{b}} = \frac{\frac{1}{1}}{\frac{a}{b}} = \frac{1}{1} \cdot \frac{b}{a} = \frac{1 \cdot b}{1 \cdot a} = \frac{b}{a}.$$

Thus,

$$\left(\frac{a}{b}\right)^{-n} = \frac{1}{\left(\frac{a}{b}\right)^n} = \frac{1}{\frac{a^n}{b^n}} = \frac{b^n}{a^n}.$$

What is $4^{1/2}$?

So, we started with exponents that were positive integers, then we added zero, then we added negatives! So now the next natural question is: what is $4^{1/2}$?

We try the same idea: what can we do to $4^{1/2}$ that, using the Basic Properties of Exponents, will allow us to get rid of the fractionary (and problematic) exponent? Here is an idea:

$$\left(4^{1/2}\right)^2 = 4^{(1/2)\cdot 2} = 4^1 = 4.$$

So, we have $\left(4^{1/2}\right)^2 = 4$. If we let $x = 4^{1/2}$ (since we don't know it), we have $x^2 = 4$. So, what is x? We have $x = 2$! Is that the only possible solution?

What is $4^{1/2}$?

We have that $x = 2$ and $x = -2$ are both solutions of $x^2 = 4$. We need to make a choice of which one we will assign for $4^{1/2}$. Since 4 is positive and every other power of 4 (so far) is positive, it makes sense to let $4^{1/2} = 2$.

What is $2^{1/2}$?

Again, there is nothing really special about 4. So, what is $2^{1/2}$? We use the same idea:

$$\left(2^{1/2}\right)^2 = 2^{(1/2)\cdot 2} = 2^1 = 2.$$

So, we have $\left(2^{1/2}\right)^2 = 2$. If we let $x = 2^{1/2}$ (since we don't know it), we have $x^2 = 2$. So, what is x? It is not an easy number, but it has a "name". It is $x = \sqrt{2}$!

Square Roots

What is $\sqrt{9}$? Is it 3 or is it ± 3? It is 3! You can ask your calculator. By definition, the square root is *always* non-negative!

The confusion comes from the fact that the solutions of $x^2 = 9$ are 3 and -3. When we solve this we have that the solutions are $\pm\sqrt{9} = \pm 3$. We need the $\pm$ symbol for the solution, since the square root only gives you the positive solution!

This is just like in the quadratic formula: the solutions for $ax^2 + bx + c = 0$ are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

If the square root "came with the $\pm$", we would not need to write it in the formula!

Computing Square Roots

What is:

- What is $\sqrt{16}$? It's 4.
- What is $\sqrt{81}$? It's 9.
- What is $\sqrt{36}$? It's 6.
- What is $\sqrt{144}$? It's 12.

But, how did you compute those? What did you do in your head?

We do not compute square roots directly: when I ask you what is $\sqrt{25}$, you ask yourself a question: what number do I square to get 25? You can remember or try a few numbers, and then you see that $\sqrt{25} = 5$, because $5^2 = 25$.

Square Roots

This is different from when I ask you to compute 7^2. You compute that directly, without asking yourself any questions! You compute $7 \cdot 7$ and get 49.

So, what is $\sqrt{2}$? It's not a nice number... (Lots of decimals!) But we know something important about it:

Whatever number $\sqrt{2}$ actually is, I know (it is positive and) if I square it I get 2!

If you really need to see it, here is an approximation:

$$\sqrt{2} = 1.4142135623730950488016887242096980786\ldots$$

What is $a^{1/2}$?

So, in general, what is $a^{1/2}$? We repeat the same idea:

$$\left(a^{1/2}\right)^2 = a^{(1/2)\cdot 2} = a^1 = a.$$

So, we have $\left(a^{1/2}\right)^2 = a$. If we let $x = a^{1/2}$ (since we don't know it), we have $x^2 = a$. So, what is x? It is $x = \sqrt{a}$! (We again choose the non-negative one!) But there is a problem! What is it? What if $a < 0$? We do not have square roots of negative numbers!

We have

$$a^{1/2} = \sqrt{a} \quad \text{if } a \geq 0.$$

(If $a < 0$, we leave $a^{1/2}$ undefined!) Note that $a^{1/2} \geq 0$!

How about $8^{1/3}$?

How about $8^{1/3}$? What should it be? We follow the same idea:

$$\left(8^{1/3}\right)^3 = 8^{(1/3)\cdot 3} = 8^1 = 8.$$

So, we have $\left(8^{1/3}\right)^3 = 8$. If we let $x = 8^{1/3}$ (since we don't know it), we have $x^3 = 8$. So, what is x? It is $x = 2$!

What is $a^{1/n}$?

So, in general, what is $a^{1/n}$?

$$\left(a^{1/n}\right)^{n} = a^{(1/n)\cdot n} = a^{1} = a.$$

So, we have $\left(a^{1/n}\right)^{n} = a$. If we let $x = a^{1/n}$ (since we don't know it), we have $x^{n} = a$. So, what is x? It is $x = \sqrt[n]{a}$!

Note that the n-th root of a, denoted by $\sqrt[n]{a}$ is a number whose n-th power is a. If $a < 0$, it might not exist, but it always exists if $a \geq 0$.

So, in general we have:

$$a^{1/n} = \sqrt[n]{a}, \quad \text{if } a \geq 0.$$

Note that $(-8)^{1/3} = \sqrt[3]{-8} = -2$, but we will keep assuming that the base a is non-negative!

What is $a^{m/n}$?

This is easy now! If we want the Basic Properties of Exponents to work, we have

$$a^{m/n} = a^{m\cdot(1/n)} = (a^m)^{1/n} = \sqrt[n]{a^m}.$$

Alternatively, we have

$$a^{m/n} = a^{m\cdot(1/n)} = a^{(1/n)\cdot m} = \left(a^{1/n}\right)^m = \left(\sqrt[n]{a}\right)^m.$$

(They are the same number!)

So, we have

$$a^{m/n} = \sqrt[n]{a^m} \text{ or } \left(\sqrt[n]{a}\right)^m \quad \text{for all } a \geq 0.$$

Summary

Theorem (Summary)

We have:

- $a^n = \underbrace{a \cdot a \cdots a}_{n \text{ factors}}$ *for any positive integer n.*
- $a^1 = a.$
- $a^0 = 1$ *for all* $a \neq 0.$
- $a^{-n} = \dfrac{1}{a^n}$ *for all* $a \neq 0.$
- $a^{m/n} = \sqrt[n]{a^m}$ *or* $\left(\sqrt[n]{a}\right)^m$ *for all* $a \geq 0.$
- $a^m \cdot a^n = a^{m+n}.$
- $(a^m)^n = a^{m \cdot n}.$

One Last Property

Here is one last property: if n is a positive integer, then

$$(a \cdot b)^n = \underbrace{ab \cdot ab \cdots ab}_{n \text{ factors of } ab} = \underbrace{a \cdot a \cdots a}_{n \text{ factors of } a} \cdot \underbrace{b \cdot b \cdots b}_{n \text{ factors of } b} = a^n \cdot b^n,$$

i.e.,

$$(a \cdot b)^n = a^n \cdot b^n.$$

Our choices for zero, negative, and fractional exponents guarantee that this property still works.

Examples

Example

What is $(32.1925)^0$?

Answer: 1.

Example

What is $\left(\frac{5}{4}\right)^3$?

Answer: $\left(\dfrac{5}{4}\right)^3 = \dfrac{5^3}{4^3} = \dfrac{125}{64}$.

Example

What is $\left(\frac{5}{4}\right)^{-2}$?

Answer: $\left(\dfrac{5}{4}\right)^{-2} = \left(\dfrac{4}{5}\right)^2 = \dfrac{4^2}{5^2} = \dfrac{16}{25}$.

Examples

Example

What is $125^{1/3}$?

Answer: $125^{1/3} = \sqrt[3]{125} = 5$.

Example

What is $\left(\frac{9}{4}\right)^{-1/2}$?

Answer: $$\left(\frac{9}{4}\right)^{-1/2} = \left(\frac{4}{9}\right)^{1/2} = \frac{4^{1/2}}{9^{1/2}} = \frac{\sqrt{4}}{\sqrt{9}} = \frac{2}{3}.$$

Examples

Example

What is $81^{3/4}$?

Answer: Remember we can either do $\sqrt[4]{81^3}$ or $\left(\sqrt[4]{81}\right)^3$. Which is best?
If we try the first, we have to compute $\sqrt[4]{531{,}441}$. What is that?!
If we try the second, we have to compute $\left(\sqrt[4]{81}\right)^3 = 3^3 = 27$.

Example

What is $\left(\frac{16}{25}\right)^{-3/2}$?

Answer: $\left(\frac{16}{25}\right)^{-3/2} = \left(\frac{25}{16}\right)^{3/2} = \frac{25^{3/2}}{16^{3/2}} = \frac{\left(\sqrt{25}\right)^3}{\left(\sqrt{16}\right)^3} = \frac{5^3}{4^3} = \frac{125}{64}$.

Missing Anything?

So, we started with positive integer exponents, added zero, then added negatives, then added fractions. Are we still missing anything?

Yes! We are missing irrational exponents! Like: $\sqrt{2}$, $\sqrt{3}$, π, e (if you don't know e, we will talk about it later), $\pi + \sqrt{5}$, etc. (There are more irrational number than rationals!)

Rational Numbers

Remember that rational numbers are fractions with integer numerator and denominator. In terms of decimal representation, a number is rational if and only if:

- the decimal expansion stops, like 1234.567,
- or it repeats the same pattern, like $0.451451451451\ldots$.

The first one is easy to see. We have:

$$1234.567 = \frac{1234567}{1000}.$$

Rational Numbers

How about the second one? Let's start with something simpler. How can we write $0.55555\ldots$ as a fraction?

$$0.55555\ldots = \frac{5}{9}.$$

Also, we have:

$$0.33333\ldots = \frac{1}{3} = \frac{3}{9} \quad \text{and} \quad 0.6666\ldots = \frac{2}{3} = \frac{6}{9}.$$

Rational Numbers

So, what is the fraction that represents $0.451451451\ldots$? It's $\frac{451}{999}$! Let's see why: let

$$\begin{aligned} x &= 0.451451451\ldots && \text{multiplying by 1000} \\ 1000x &= 451.451451451\ldots && \text{subtracting the first from the second} \\ 999x &= 451 && \text{solving for } x \\ x &= \frac{451}{999} && \text{putting the first and last together} \end{aligned}$$

$$0.451451\ldots = \frac{451}{999}.$$

This works in general. For instance,

$$0.123456123456123456\ldots = \frac{123456}{999999}.$$

About 0.9999 . . .

So, we have that

$$0.9999\ldots = \frac{9}{9} = 1!$$

That can't be right! (Can it?)

Well, $1 - 0.99999\ldots = 0.00000\ldots = 0$, so $1 = 0.99999\ldots$.

If you are still not convinced, $\frac{1}{3} = 0.33333\ldots$, so multiplying by 3, we have $\frac{3}{3} = 0.999999\ldots$, so $1 = 0.9999\ldots$.

Indeed, it is true that

$$0.9999\cdots = 1.$$

Example

Example

Convert 12.1234567567567 . . . to a fraction.

$$\begin{aligned} 12.1234567567567\ldots &= 12.1234 + 0.0000567567567\ldots \\ &= \frac{121234}{10000} + \frac{1}{10000} \cdot 0.567567567\ldots \\ &= \frac{121234}{10000} + \frac{1}{10000} \cdot \frac{567}{999} \\ &= \frac{999 \cdot 121234 + 567}{9990000} \\ &= \frac{4485679}{370000}. \end{aligned}$$

Irrational Numbers

So, our irrational numbers have decimal expansions that never end and repeat no pattern:

$$\sqrt{2} = 1.4142135623730950488016887242 0\ldots$$
$$\pi = 3.1415926535897932384626433832 7\ldots$$
$$e = 2.7182818284590452353602874713 5\ldots$$
$$\sqrt{2} + \pi = 4.5558062159628882872643321074 8\ldots$$

What is 2^π?

So, what is 2^π? Well, we will give up on that one... To really answer it we would need Calculus! (Limits, to be precise.)

What we will do is approximate:

$$2^\pi \approx 2^{3.14159} = 2^{\frac{314159}{100000}} = \left(\sqrt[100000]{2}\right)^{314159} \approx 8.824961595059\ldots.$$

The better is our approximation of π, the better is our approximation of 2^π.