Algebra II

Fractions

Project GRAD SI 2022

THE UNIVERSITY OF TENNESSEE KNOXVILLE

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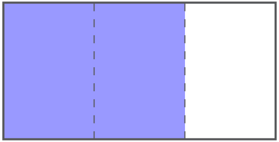
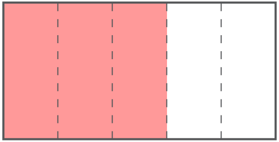
Fractions

The main idea of a fraction is quite simple: parts of a whole.
For example, we can visualize $\frac{2}{3}$ as:



And $\frac{3}{5}$ as:

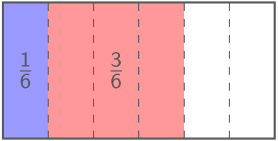


Addition

The *bottom number* is called the denominator and represents in how many pieces we divide the whole. The denominator can never be 0!

The *top number* is called the numerator and represents how many pieces with pick.

Adding with the same denominator is easy, for instance $\frac{1}{6} + \frac{3}{6}$:



So: $\frac{1}{6} + \frac{3}{6} = \frac{4}{6}$.

Addition

So, in general we have:

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}.$$

Example

Compute $\frac{3}{111} + \frac{17}{111}$.

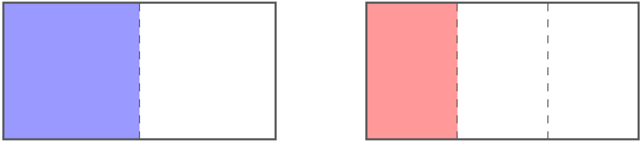
Solution.

$$\frac{3}{111} + \frac{17}{111} = \frac{17+3}{111} = \frac{20}{111}.$$

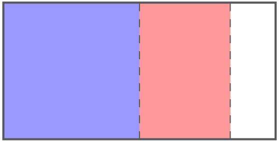
□

Addition

The problem comes when we have different denominators. For instance, what is $\frac{1}{2} + \frac{1}{3}$?



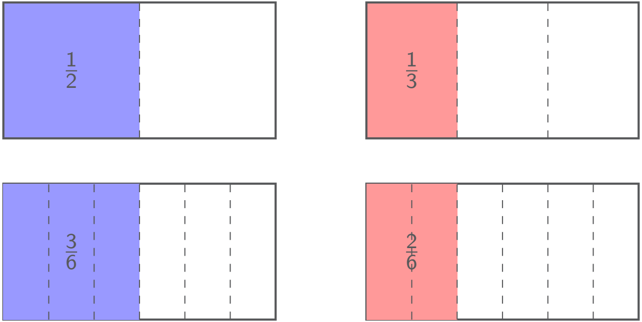
So, $\frac{1}{2} + \frac{1}{3}$ is:



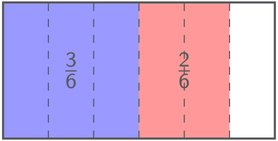
Problem: Like different units.

Addition

To fix that, we find a common unit:



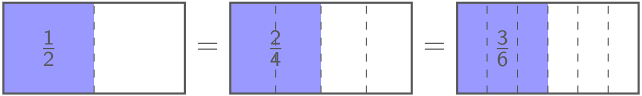
Then:



So: $\frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6} = \frac{5}{6}$.

Equality

As we can see in the example above, many different fractions can represent the same quantity:



In general:

$$\frac{a}{b} = \frac{k \cdot a}{k \cdot b} \text{ for all } k \neq 0.$$

Equality

In general, perhaps an easier way to check the equality of two fractions $\frac{a}{b}$ and $\frac{c}{d}$ is:

$$\frac{a}{b} = \frac{c}{d} \quad \text{if and only if} \quad ad = cb.$$

Example

Are $\frac{6}{10}$ and $\frac{21}{35}$ equal?

Solution.

We have $6 \cdot 35 = 210$ and $10 \cdot 21 = 210$, so yes, they are equal. □

Equality

Example

Are $\frac{8}{14}$ and $\frac{20}{34}$ equal?

Solution.

We have $8 \cdot 34 = 272$ and $20 \cdot 14 = 280$, so no, they are *not* equal. □

Note: Which one is larger, then? One can show that since the fractions are positive and $8 \cdot 34 = 272 < 280 = 20 \cdot 14$, we have $\frac{8}{14} < \frac{20}{34}$. We simply multiply both fractions by $14 \cdot 34$! (We can also write the two fractions with a common denominator.)

Addition

So, to add $\frac{a}{b} + \frac{c}{d}$ is general, we need to rewrite the fractions so that they have a common denominator. One number that can be easily seen to be a common multiple of both b and d is bd. (It might not be the least common multiple, but it *does* work nonetheless!)

We then have:

$$\frac{a}{b} + \frac{c}{d} = \frac{a \cdot d}{b \cdot d} + \frac{c \cdot b}{d \cdot b} = \frac{a \cdot d + c \cdot b}{b \cdot d}.$$

In other words:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}.$$

Addition

Example

Compute $\frac{1}{3} + \frac{2}{5}$.

Solution.

We have

$$\frac{1}{3} + \frac{2}{5} = \frac{1 \cdot 5}{3 \cdot 5} + \frac{2 \cdot 3}{5 \cdot 3} = \frac{5}{15} + \frac{6}{15} = \frac{11}{15}.$$

□

Addition

Example

Compute $\frac{3}{8} + \frac{5}{6}$.

Solution.

We have

$$\frac{3}{8} + \frac{5}{6} = \frac{3 \cdot 6}{8 \cdot 6} + \frac{5 \cdot 8}{6 \cdot 8} = \frac{18}{48} + \frac{40}{48} = \frac{58}{48}.$$

Alternatively, using the least common multiple,

$$\frac{3}{8} + \frac{5}{6} = \frac{3 \cdot 3}{8 \cdot 3} + \frac{5 \cdot 4}{6 \cdot 4} = \frac{9}{24} + \frac{20}{24} = \frac{29}{24}.$$

□

Subtraction

Subtraction works pretty much the same way. So we have:

$$\frac{a}{c} - \frac{b}{c} = \frac{a - b}{c},$$

and

$$\frac{a}{b} - \frac{c}{d} = \frac{a \cdot d}{b \cdot d} - \frac{c \cdot b}{b \cdot d} = \frac{ad - bc}{bd}.$$

Multiplication

We interpret $a \cdot \frac{1}{b}$ as $a \div b$, which is the same as $\frac{a}{b}$.

So, $\frac{1}{b} \cdot \frac{1}{d} = 1 \cdot \frac{1}{b} \cdot \frac{1}{d}$ is the same as dividing 1 into b pieces, and then each of these smaller pieces into d pieces. This is the same as dividing 1 into $b \cdot d$ pieces. Hence,

$$\frac{1}{b} \cdot \frac{1}{d} = \frac{1}{b \cdot d}.$$

Then,

$$\frac{a}{b} \cdot \frac{c}{d} = \left(a \cdot \frac{1}{b}\right) \cdot \left(c \cdot \frac{1}{d}\right) = (a \cdot c) \cdot \left(\frac{1}{b} \cdot \frac{1}{d}\right) = (a \cdot c) \cdot \frac{1}{b \cdot d} = \frac{a \cdot c}{b \cdot d}.$$

Hence $\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$.

Multiplication

Example

Compute $\frac{1}{3} \cdot 8$.

Proof.

We have $\frac{1}{3} \cdot 8 = \frac{1}{3} \cdot \frac{8}{1} = \frac{1 \cdot 8}{3 \cdot 1} = \frac{8}{3}$. □

Example

Compute $\frac{3}{7} \cdot \frac{2}{5}$.

Proof.

We have $\frac{3}{7} \cdot \frac{2}{5} = \frac{3 \cdot 2}{7 \cdot 5} = \frac{6}{35}$. □

Division

Division is the opposite of multiplication. (One cancels the other.) So, $a \div \frac{1}{b}$ is, instead of dividing a by b, multiplying a by b.

Hence,

$$a \div \frac{1}{b} = \frac{a}{1/b} = a \cdot b.$$

With similar ideas, we obtain the more general formula:

$$\frac{a/b}{c/d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{a \cdot d}{b \cdot c}.$$

Division

Example

Compute $\frac{2/5}{1/3}$.

Proof.

We have $\frac{2/5}{1/3} = \frac{2}{5} \cdot \frac{3}{1} = \frac{6}{5}$. □

Example

Compute $\frac{3/2}{7}$.

Proof.

We have $\frac{3/2}{7} = \frac{3/2}{7/1} = \frac{3}{2} \cdot \frac{1}{7} = \frac{3}{14}$. □