Algebra II

Logarithms

Project GRAD SI 2022

THE UNIVERSITY OF TENNESSEE KNOXVILLE

June 15, 2022

Preface

We now turn our attention to logarithms, or logs, for short. But before we get into it, let me ask you some simple questions about exponents:

$$\text{If } 2^x = 8, \text{ then } x = 3.$$

$$\text{If } 3^x = 81, \text{ then } x = 4.$$

$$\text{If } 5^x = \frac{1}{25}, \text{ then } x = -2.$$

$$\text{If } 16^x = 2, \text{ then } x = 1/4.$$

That was not too bad!

Logs

We've just computed four examples of logs!

Definition

The log base b of a, (with $a, b > 0$) which we write as $\log_b(a)$, is the power of b that gives a. In other words, $\log_b(a)$ is the number x such that $b^x = a$.

(Similar to square roots, we have to solve an equation to compute these logs!) So:

- Since the x for which $2^x = 8$ was $x = 3$, we have that $\log_2(8) = 3$.
- Since the x for which $3^x = 81$ was $x = 4$, we have that $\log_3(81) = 4$.
- Since the x for which $5^x = \frac{1}{25}$ was $x = -2$, we have that $\log_5(\frac{1}{25}) = -2$.
- Since the x for which $16^x = 2$ was $x = 1/4$, we have that $\log_{16}(2) = 1/4$.

Examples

Example

Compute $\log_8(64)$.

Solution: We ask what is the x such that $8^x = 64$. We get $x = 2$. So, $\log_8(64) = 2$.

Example

Compute $\log_2(16)$.

Solution: We ask what is the x such that $2^x = 16$. We get $x = 4$. So, $\log_2(16) = 4$.

Example

Compute $\log_\pi(1)$.

Solution: We ask what is the x such that $\pi^x = 1$. We get $x = 0$. So, $\log_\pi(1) = 0$.

Examples

Example

Compute $\log_3(\frac{1}{9})$.

Solution: We ask what is the x such that $3^x = \frac{1}{9}$. We get $x = -2$. So, $\log_3(\frac{1}{9}) = -2$.

Example

Compute $\log_{2/3}(\frac{27}{8})$.

Solution: We ask what is the x such that $\left(\frac{2}{3}\right)^x = \frac{27}{8}$. We get $x = -3$. So, $\log_{2/3}(\frac{27}{8}) = -3$.

Examples

Example

Compute $\log_{216}(6)$.

Solution: We ask what is the x such that $216^x = 6$. We get $x = 1/3$. So, $\log_{216}(6) = 1/3$.

Example

Compute $\log_{4/9}(\frac{3}{2})$.

Solution: We ask what is the x such that $\left(\frac{4}{9}\right)^x = \frac{3}{2}$. We get $x = -\frac{1}{2}$. So, $\log_{4/9}(\frac{3}{2}) = -\frac{1}{2}$.

Examples

Example

Compute $\log_{27}(81)$.

Solution: We ask what is the x such that $27^x = 81$. That's harder. Let's try some numbers:

$$27^1 = 27, \text{so } x = 1 \text{ is too small.}$$

$$27^2 = 729, \text{so } x = 2 \text{ is too large.}$$

$$27^{3/2} = \left(\sqrt{27}\right)^3 = 140.2961154\ldots, \text{so } x = 3/2 \text{ is too large.}$$

$$27^{5/4} = \left(\sqrt[4]{27}\right)^5 = 61.54669\ldots, \text{so } x = 5/4 \text{ is too small.}$$

Examples

Example

Compute $\log_{27}(81)$.

Solution: We ask what is the x such that $27^x = 81$. It seems that trial and error is not going to work... We need a better idea!

Do you recognize both 27 and 81 as powers of another (same) number? We have $27 = 3^3$ and $81 = 3^4$! (Both are powers of 3!) So:

$$27^x = 81 \implies (3^3)^x = 3^4 \implies 3^{3x} = 3^4 \implies 3x = 4 \implies x = \frac{4}{3}.$$

So, $\log_{27}(81) = \frac{4}{3}$. This is a good trick to remember! (We will use this idea again later!)

Examples

Example

What is $\log_3(4)$?

Solution: We need to solve $3^x = 4$. That's hard! In fact, it is not a "nice" number at all. With a little trial and error we could get close, but not exactly.

This is similar to $\sqrt{2}$: it is a complicated number that we can compute to some accuracy with some trial and error. The calculator tells me

$$\log_3(4) = 1.2618595071429148741990542286855217086\ldots$$

We will talk about using a calculator with logs later.

Some First Properties of Logs

Example

What is $b^{\log_b(a)}$? (For example, what is $3^{\log_3(4)}$?)

Solution: Remember: by definition, $\log_b(a)$ is the exponent of b that gives a. So, $b^{\log_b(a)} = a$! (So, we have that $3^{\log_3(4)} = 4$.)

This is similar to how we might not know what $\sqrt{2}$ is, but we do know that $(\sqrt{2})^2 = 2$!

Some First Properties of Logs

Example

What is $\log_b(b^a)$?

Solution: We have, by definition, that $\log_b(b^a)$ is the power of b that gives b^a. So, $\log_b(b^a) = a$!

Some First Properties of Logs

Theorem

We have:

- $b^{\log_b(a)} = a$,
- $\log_b(b^a) = a$.

These are not really *properties* of the log, as much as it is its *definition*. In "fancy" math terminology, these amount to saying that the functions b^x and $\log_b(x)$ are inverses of each other: whatever one does to some number a, the other "undoes" it.

Again, this is similar to the n-th root: whatever x^n does to some number a, $\sqrt[n]{x}$ undoes it: $\sqrt[n]{a^n} = a$ and $(\sqrt[n]{a})^n = a$, if $a > 0$.

Some Examples

Example

Compute $\log_{13}(13^7)$.

Solution: $\log_{13}(13^7) = 7$.

Example

Compute $5^{\log_5(2)}$.

Solution: $5^{\log_5(2)} = 2$.

Basic Properties of Logs

Note that logs and exponents are closely related: logs always ask what is the exponent. Therefore, the Basic Properties of Exponents yield log properties!

Let's review the Basic Properties of Exponents:

Theorem (Basic Properties of Exponents)

If b is a positive real number and m and n are (any) real numbers, then:

1. $b^m \cdot b^n = b^{m+n}$;
2. $(b^m)^n = b^{m \cdot n}$.

Basic Properties of Logs

What is $\log_b(xy)$? Suppose that $x = b^m$ and $y = b^n$. Then, the question becomes what is $\log_b(b^m \cdot b^n)$.

But the Basic Properties of Exponent tell us that $b^m \cdot b^n = b^{m+n}$. So, we get $\log_b(b^m \cdot b^n) = \log_b(b^{m+n})$.

But what is $\log_b(b^{m+n})$? From what we've just seen, $\log_b(b^{m+n}) = m + n$! So, if $x = b^m$ and $y = b^n$, then $\log_b(xy) = m + n$.

Is there an m such that $b^m = x$? *Yes!* From what we've just seen, we have that $b^{\log_b(x)} = x$! In other words, $m = log_b(x)$ works! Similarly, we have that $y = b^{\log_b(y)}$, so $n = \log_b(y)$ works!

Basic Properties of Logs

So, we have:

$$\begin{aligned}\log_b(xy) &= \log_b(b^{\log_b(x)} \cdot b^{\log_b(y)}) \\ &= \log_b(b^{\log_b(x)+\log_b(y)}) \\ &= \log_b(x) + \log_b(y).\end{aligned}$$

So, we have the First Basic Property of Logs:

$$\log_b(xy) = \log_b(x) + \log_b(y).$$

(Note that it comes from the First Basic Property of Exponents: $b^m \cdot b^n = b^{m+n}$.)

Basic Properties of Logs

Now, what is $\log_b(x^y)$? We take a similar approach: let's write $x = b^{\log_b(x)}$. Then, we have:

$$\begin{aligned}\log_b(x^y) &= \log_b\left(\left(b^{\log_b(x)}\right)^y\right) \\ &= \log_b\left(b^{\log_b(x)\cdot y}\right) \\ &= \log_b(x)\cdot y = y\cdot\log_b(x)\end{aligned}$$

So, we have the Second Basic Property of Logs:

$$\log_b(x^y) = y\cdot\log_b(x).$$

(Note that it comes from the Second Basic Property of Exponents: $(b^m)^n = b^{m\cdot n}$.)

Basic Properties of Logs

In summary, we have:

Theorem (Basic Properties of Logs)

If b, x, and y are all positive real numbers, then:

1. $\log_b(x \cdot y) = \log_b(x) + \log_b(y)$;
2. $\log_b(x^y) = y \cdot \log_b(x)$.

Note that:

$$\log_b\left(\frac{x}{y}\right) = \log_b(x \cdot y^{-1}) = \log_b(x) + \log_b(y^{-1}) = \log_b(x) - \log_b(y).$$

So:

$$\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y).$$

Computing Logs

Earlier, I asked you what $\log_3(4)$ was. I told you it was not a "nice" number, and that I computed it to be:

$$\log_3(4) = 1.2618595071429148741990542286855217086\ldots.$$

Of course, I used a calculator. But, if you try to use you calculator, you will see only one button for log:

Computing Logs

But the key just says "LOG". What is the base? By (engineering) convention, log without a base specified means base 10! So $\log = \log_{10}$. (This is similar to how $\sqrt{x} = \sqrt[2]{x}$.)

In fact, your calculator does have another log:

The key "LN" stands for *natural log*. The natural log is the log base e, where $\mathrm{e} = 2.7182\ldots$ is this "mysterious" number I keep mentioning (that will show up in finances later). So, $\ln = \log_{\mathrm{e}}$.

Change of Base

So the calculator has only two logs: $\log = \log_{10}$ and $\ln = \log_{\mathrm{e}}$. So, how do we compute $\log_3(4)$? So, we need to change the base (from 3 to either 10 or e)!

Here is the idea to change from base b to base c:

$$\begin{aligned} \log_c(x) &= \log_c(b^{\log_b(x)}) \\ &= \log_b(x) \cdot \log_c(b). \end{aligned}$$

Solving for $\log_b(x)$, gives:

Theorem (Change of Base Formula)

$$\log_b(x) = \frac{\log_c(x)}{\log_c(b)}.$$

Computing Logs

So, how do we compute $\log_3(4)$ with a calculator? We change to a log that the calculator can compute:

$$\log_3(4) = \frac{\log(4)}{\log(3)} \approx \frac{0.602059991}{0.477121254} \approx 1.261859507,$$

or

$$\log_3(4) = \frac{\ln(4)}{\ln(3)} \approx \frac{1.386294361}{1.098612288} \approx 1.261859507.$$

Standard Logs

Engineers like logs base 10 because they give the order of magnitude of the argument. In other words, it gives how many digits it has. If you round up the number of value of the log, you get the number of digits:

number	log	number of digits
133	$2.12385\ldots$	3
2590	$3.41329\ldots$	4
1234567	$6.09151\ldots$	7
2^{100}	$30.10299\ldots$	31

(Note that we compute this last one as $100 \cdot \log(2)$.)

Standard Logs

The reason for this is simple: a number x has n digits if and only if $10^{n-1} \leq x < 10^n$, as

$$10^k = 1\underbrace{00\cdots0}_{k\text{ zeros}},$$

the smallest number with k digits.

So, if x has n digits, taking logs we have

$$\log(10^{n-1}) \leq \log(x) < \log(10^n),$$

or

$$n - 1 \leq \log(x) < n.$$

Standard Logs

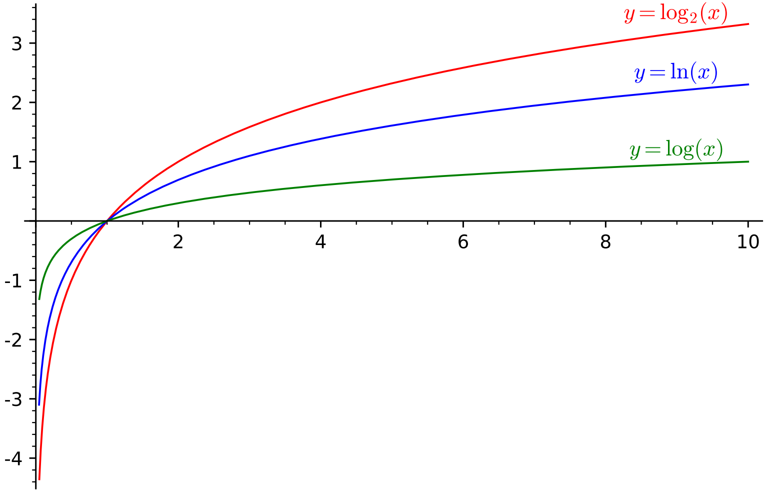
On the other hand, Computer Scientist/Engineers prefer $\log_2$, since computer work with binary numbers. Because of that, in some advanced computer books and papers, they use log (with no base written) to represent $\log_2$ instead of $\log_{10}$.

Also, mathematicians prefer $\ln = \log_{\mathrm{e}}$, since it has “nicer mathematical properties”. So, also in some more advanced math books and papers, we write log for $\log_{\mathrm{e}}$.

In finances, also the log base e is preferred, but as far as I know, they just write ln (and not log).

Graphs

Here are the graphs for $\log_2$, ln, and log:



Solving Equations

Solving equations using logs will be crucial in finances!

Similarly to how the square root was introduce to solve quadratic equations, logs were introduced to solve equations with the unknown x in the exponent.
For example, how do we solve:

$$3^x = 4?$$

If you remember your properties of logs, it's clear that $x = \log_3(4)$. If we actually want to compute this, we change to base 10 or e and use the calculator, as we've done above.

Solving Equations

Here is another way to do this. Remember this: when solving equations with unknown in the exponent, *take logs from both sides*!

Which log do you use? It's better to use one that you can compute, so either log or ln.

In our previous example, we can do:

$$3^x = 4 \implies \log(3^x) = \log(4) \implies$$
$$x \cdot \log(3) = \log(4) \implies x = \frac{\log(4)}{\log(3)}.$$

No need to change base!

Solving Equations

Example

Solve $5^x = 7$ for x.

Solution: We take logs from both sides. (Let's use ln this time.)

$$5^x = 7 \quad \Longrightarrow \quad \ln(5^x) = \ln(7) \quad \Longrightarrow$$

$$x \cdot \ln(5) = \ln(7) \quad \Longrightarrow \quad x = \frac{\ln(7)}{\ln(5)} = 1.20906\ldots.$$

Solving Equations

Example

Solve $5 \cdot 3^{2x^2} = 19$ for x.

Solution: We take logs from both sides.

$$5 \cdot 3^{2x^2} = 19 \implies \log(5 \cdot 3^{2x^2}) = \log(19) \implies$$

$$\log(5) + \log(3^{2x^2}) = \log(19) \implies \log(3^{2x^2}) = \log(19) - \log(5) \implies$$

$$2x^2 \log(3) = \log(19) - \log(5) \implies x^2 = \frac{\log(19) - \log(5)}{2\log(3)} \implies$$

$$x = \pm\sqrt{\frac{\log(19) - \log(5)}{2\log(3)}} = \pm 0.77947749\ldots.$$