

Algebra II

The Power of Powers

Project GRAD SI 2022



June 15, 2022

A Story

A long time ago, the king of England was bored. He asked the wisest man in court to find him something to do. The wise man showed the king chess!



(He did not invent it, he just showed it to the king.)

A Story

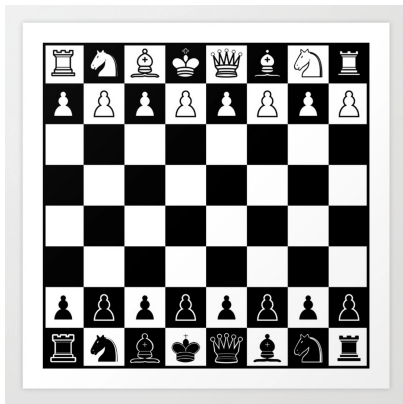
The king liked it so much he told the man he could ask for whatever he wanted!

The man asked for the following: start with **1** grain of corn in the first square of the chessboard. In the second square, put **2** grains. In the next, **4** grains. Double it again, putting **8** grains in the next one. And, **doubling** like that, fill the whole chessboard with corn.

The king was puzzled: “All you want is some corn?”

How much corn?

How many grains of corn were on the last square? The chessboard is 8×8 :



So, there are **64** squares on the board.

How much corn?

How many grains of corn were on the last square?

We have:

- Square 1: 1 grain.
- Square 2: 2 grains.
- Square 3: 4 grains.
- Square 4: 8 grains.
- Square 5: 16 grains.
- Square 6: 32 grains.
- \vdots
- Square 64: ?? grains.

It would takes us sometime to keep doubling...

How much corn?

So, let's take a *shortcut*. We have:

- Square 1: 1 grain.
- Square 2: $2 \cdot 1 = 2 = 2^1$ grains.
- Square 3: $2 \cdot 2 = 2^2 = 4$ grains.
- Square 4: $2 \cdot 2^2 = 2^3 = 8$ grains.
- Square 5: $2 \cdot 2^3 = 2^4 = 16$ grains.
- \vdots
- Square 64: $2^{64-1} = 2^{63}$ grains.

Therefore, we have 2^{63} (whatever the number is) grains of corn on the last square!

How much corn?

How much corn we have **in total**? Well, it is

$$1 + 2^1 + 2^2 + 2^3 + \dots + 2^{63}$$

grains of corn. Can we get a better idea of what this number actually is?

Let's look for a pattern again!

How much corn?

Let's see:

- One term: 1 .
- Two terms: $1 + 2 = 3$.
- Three terms: $1 + 2 + 2^2 = 7$.
- Four terms: $1 + 2 + 2^2 + 2^3 = 15$.
- Five terms: $1 + 2 + 2^2 + 2^3 + 2^4 = 31$.
- Six terms: $1 + 2 + 2^2 + 2^3 + 2^4 + 2^5 = 63$.
- \vdots
- Sixty four terms: $1 + 2 + 2^2 + 2^3 + \cdots + 2^{63} = ??$.

How much corn?

Let's see:

- One term: $1 = 2 - 1$.
- Two terms: $1 + 2 = 3 = 4 - 1$.
- Three terms: $1 + 2 + 2^2 = 7 = 8 - 1$.
- Four terms: $1 + 2 + 2^2 + 2^3 = 15 = 16 - 1$.
- Five terms: $1 + 2 + 2^2 + 2^3 + 2^4 = 31 = 32 - 1$.
- Six terms: $1 + 2 + 2^2 + 2^3 + 2^4 + 2^5 = 63 = 64 - 1$.
- \vdots
- Sixty four terms: $1 + 2 + 2^2 + 2^3 + \cdots + 2^{63} = ??$.

How much corn?

Let's see:

- One term: $1 = 2 - 1$.
- Two terms: $1 + 2 = 3 = 2^2 - 1$.
- Three terms: $1 + 2 + 2^2 = 7 = 2^3 - 1$.
- Four terms: $1 + 2 + 2^2 + 2^3 = 15 = 2^4 - 1$.
- Five terms: $1 + 2 + 2^2 + 2^3 + 2^4 = 31 = 2^5 - 1$.
- Six terms: $1 + 2 + 2^2 + 2^3 + 2^4 + 2^5 = 63 = 2^6 - 1$.
- ⋮
- Sixty four terms: $1 + 2 + 2^2 + 2^3 + \cdots + 2^{63} = 2^{64} - 1$.

So, we have $2^{64} - 1$ grains of corn in the whole chessboard.

Quarters

Imagine that instead of corn, the man had asked for **quarters**. How tall do you think the last pile on the last square is? (So, it is a pile of 2^{63} **quarters**.) Is it:

- Taller than Ayres Hall on campus? *Yes!*
- Taller than the tallest building we have downtown (Knoxville)? *Yes!*
- Taller than the Empire States building (1,454 feet)? *Yes!*
- Taller than the tallest building in the world (Burj Khalifa: 2,717 feet)? *Yes!*
- Taller than Mount Everest (29,029 feet)? *Yes!*
- Does it reach the moon (238,900 miles)? *Yes!*
- Does it reach the sun (93.905 million miles)? *Yes!*
- If put in the sun, does it reach the Pluto (3.7 billion miles)? *Yes!*

And it passes Pluto by *a lot!*

Computations

Let's do some computations: the quarter is 1.75 millimeters thick. (See <https://www.usmint.gov/learn/coin-and-medal-programs/coin-specifications>)

Also,

$$2^{63} = 9,223,372,036,854,775,808.$$

- So, the pile is 18,446,744,073,709,551,616 millimeters tall.
- 1 millimeter is 0.00328084 foot. So: 52,955,713,848,440,589.66 feet tall.
- 1 foot is 0.000189394 mile. So: 10,029,494,468,611.56 miles tall.
- 1 mile is 0.00000000000017 light years. So: 1.7 light years tall.

The second nearest star to Earth, *Proxima Centauri*, is 4.24 light years from earth.

A Story

So, *all* the corn ($2^{64} - 1$ grain) is twice the pile in the last square (well, one grain less than twice). That is enough corn to cover *all of England* to height of 12 meters (or 39.37 feet).

The story goes that the king had the wise man **decapitated**. (Maybe not so wise, after all. . .)

Paper

A sheet of paper is between 0.05 and 0.1 millimeters thick. Let's say 0.05 millimeters. Suppose you fold the paper (the *whole* paper) in half 16 times. How thick will that be?

Each time you fold, you double the thickness. So, it will be

$$(0.05 \cdot 2^{16}) \text{ mm} = 3,276.8 \text{ mm} = 10.72 \text{ feet},$$

much taller than LeBron James (6'9").

On the other hand, the (top) area of a letter sheet (8.5 by 11 inches) is divided by 2 each time you fold. So, after 16 folds the area is

$$\frac{8.5 \cdot 11}{2^{16}} \text{ sq in} = 0.00142 \text{ sq in} = 0.9 \text{ sq mm}.$$

That's barely visible!