**COMPLEX EIGENVALUES OF REAL MATRICES**

The characteristic polynomial of an $n \times n$ matrix $A$ is the degree $n$ polynomial in one variable $\lambda$:

$$p(\lambda) = \det(\lambda I - A);$$

its roots are the eigenvalues of $A$. For example, in the $2 \times 2$ case, the eigenvalues are the roots of the quadratic equation:

$$\lambda^2 - tr(A)\lambda + det(A) = 0,$$

where the 'trace' $tr(A)$ is the sum of diagonal entries. If the discriminant of this quadratic equation $\Delta = tr(A)^2 - 4det(A)$ is negative, the roots (given by the quadratic formula) will be complex:

$$\lambda = a \pm ib, \text{ where } tr(A) = 2a, \quad det(A) = a^2 + b^2.$$

**Example.**

$$A = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$$

The characteristic equation is $p(\lambda) = \lambda^2 - 2\lambda + 5 = 0$, with roots $\lambda = 1 \pm 2i$.

That the two eigenvalues are complex conjugate to each other is no coincidence. *If the $n \times n$ matrix $A$ has real entries, its complex eigenvalues will always occur in complex conjugate pairs.* Thus, say, if a $7 \times 7$ matrix with real entries has $8, 3+2i, -2-5i$ and $-3i$ as eigenvalues, we know that automatically the remaining eigenvalues are $3-2i, -2+5i$ and $3i$. This is very easy to see; recall that if an eigenvalue is complex, its eigenvectors will in general be vectors with complex entries (that is, vectors in $\mathbb{C}^n$, not $\mathbb{R}^n$). If $\lambda \in \mathbb{C}$ is a complex eigenvalue of $A$, with a non-zero eigenvector $v \in \mathbb{C}^n$, by definition this means:

$$Av = \lambda v, v \neq 0.$$

Taking complex conjugates of this equation, we obtain:

$$A\bar{v} = \bar{A}\bar{v} = \bar{\lambda}\bar{v},$$

where the first equality follows from the fact that $A$ has real entries (so $\bar{A} = A$). But this means exactly that $\bar{\lambda}$ is an eigenvalue of $A$, with $\bar{v}$ as an eigenvector.

Continuing the **example** above, we find the eigenspace for the eigenvalue $1 + 2i$ by solving the homogeneous system:

$$\left\{ \begin{array}{llr} 3x_1 & -2x_2 = & (1+2i)x_1 \\ 4x_1 & -x_2 = & (1+2i)x_2 \end{array} \right. ,$$

which is the same as:

$$\left\{ \begin{array}{rrr} (2-2i)x_1 & -2x_2 = & 0 \\ 4x_1 & -(2+2i)x_2 = & 0 \end{array} \right.$$

Note that the second equation is just the first multiplied by $1+i$; the system is redundant, as it must always be if $\lambda$ is an eigenvalue (otherwise it would only have the zero solution). To get the general solution, we set $x_1 = t$ and find (say, from the first equation) $x_2 = t(1-i)$; but remember we are now working in $\mathbb{C}^2$ (not in $\mathbb{R}^2$), so $t$ is an arbitrary *complex* number. The eigenspace for $\lambda = 1 + 2i$ is the subspace of $\mathbb{C}^2$:

$$E(1+2i) = \{t \begin{bmatrix} 1 \\ 1-i \end{bmatrix}; t \in \mathbb{C}\}.$$

Now, it follows from the comments above that the eigenspace for the complex conjugate eigenvalue $\bar{\lambda} = 1 - 2i$ can be written down with *no further calculation*, simply by taking complex conjugates! It is:

$$E(1-2i) = \{t \begin{bmatrix} 1 \\ 1+i \end{bmatrix}; t \in \mathbb{C}\}.$$

**Interpretation in $\mathbb{R}^2$.** Recall that we start with a real $n \times n$ matrix, which in principle 'acts' on vectors in $\mathbb{R}^n$ (for now, we concentrate on $n = 2$.) It would be more satisfying (and is also important for applications) if we could somehow "interpret complex eigenvalues in real terms". This would be analogous to the interpretation of complex multiplication in terms of vectors in $\mathbb{R}^2$. Recall that if we write a complex number $\lambda$ in 'polar form':

$$\lambda = r(\cos\theta + i\sin\theta), \quad r = |\lambda|,$$

and identify a second complex number $z = u+iv$ with the real vector $\begin{bmatrix} u \\ v \end{bmatrix}$, then since the real and imaginary parts of the product $\lambda z$ are:

$$Re(\lambda z) = r(u\cos\theta - v\sin\theta), \quad Im(\lambda z) = r(u\sin\theta + v\cos\theta),$$

we should likewise ‘identify’ the complex number $\lambda z$ with the vector:

$$\begin{bmatrix} Re(\lambda z) \\ Im(\lambda z) \end{bmatrix} = r \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = rR_\theta \begin{bmatrix} u \\ v \end{bmatrix},$$

where $R_\theta$ is the $2 \times 2$ matrix that implements rotation by $\theta$ radians in $\mathbb{R}^2$ (counterclockwise). This means that (in the way it acts on $\mathbb{R}^2$, identified with $\mathbb{C}$ in the usual way) complex multiplication by $\lambda$ can be thought of as counterclockwise rotation by $\theta$, followed by expansion by a factor $r = |\lambda|$. We seek something analogous to this for a complex eigenvalue-eigenvector pair of a real $2 \times 2$ matrix $A$.

Return to the defining equation for an eigenvalue-eigenvector pair: $Av = \lambda v$. With $v = Re(v) + iIm(v)$ and $\lambda = a + ib$ (where the vectors $Re(v)$ and $Im(v)$, the real and imaginary parts of $v$, are in $\mathbb{R}^2$), the real and imaginary parts of this equation read, respectively:

$$ARe(v) = aRe(v) - bIm(v);$$

$$AIm(v) = bRe(v) + aIm(v).$$

We can combine these two vector equations into a single matrix equation, if we define the real $2 \times 2$ matrix $V$ by:

$$V = [Re(v)|Im(v)] \text{ (by columns).}$$

Then the right-hand sides of the two equations above are, respectively, the vectors:

$$V \begin{bmatrix} a \\ -b \end{bmatrix} \text{ and } V \begin{bmatrix} b \\ a \end{bmatrix}.$$

Recall how matrix multiplication works: the product matrix $AV$ is the $2 \times 2$ matrix with column vectors $ARe(v)$ and $AIm(v)$, that is, with the two vectors just computed as its column vectors. But the $2 \times 2$ matrix with these vectors as column vectors can also be written as a matrix product:

$$[V \begin{bmatrix} a \\ -b \end{bmatrix} | V \begin{bmatrix} b \\ a \end{bmatrix}] = V \begin{bmatrix} a & b \\ -b & a \end{bmatrix}.$$

To summarize, defining the $2 \times 2$ real matrix $\Lambda$ by:

$$\Lambda = \begin{bmatrix} a & b \\ -b & a \end{bmatrix},$$

we have shown that the complex eigenvector equation is equivalent to the equation for real $2 \times 2$ matrices:

$$AV = [ARe(v)|AIm(v)] = V\Lambda.$$

Going back to our current **example**, with $\lambda = 1 + 2i$ and $v = (1, 1 - i)$, this identity reads:

$$\begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$

(check directly that this is true!)

By writing the eigenvalue in polar form as $\lambda = |\lambda|(\cos\theta + i\sin\theta)$, we can express the matrix $\Lambda$ in a more suggestive form:

$$\Lambda = |\lambda| \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = |\lambda| R_{(-\theta)},$$

the product of a positive number (an expansion factor) and a *clockwise* rotation matrix.

A final point of view on this: $\Lambda$ is known as the 'real standard form' of the $2 \times 2$ matrix $A$ with complex eigenvalues $a \pm ib$. In what sense is it the 'standard form' of $A$? The matrix identity above can be written in a form reminiscent of the 'change of basis formula':

$$V^{-1}AV = \Lambda$$

(we say $\Lambda$ and $A$ are 'conjugate' via $V$). What is the basis involved? It is not hard to see that $\mathcal{B} = \{v_1 = Re(v), v_2 = Im(v)\}$ is a basis of $\mathbb{R}^2$ (if $v_1$ and $v_2$ were linearly dependent, $v$ and $\bar{v}$ would be l.d. also, which is impossible for eigenvectors with different eigenvalues). Above we argued that $A$ acts on this basis in a simple way:

$$Av_1 = av_1 - bv_2, \quad Av_2 = bv_1 + av_2,$$

which means that the linear transformation $T$ of $\mathbb{R}^2$ with matrix given by $A$ in the standard basis is given in the basis $\mathcal{B}$ by $\Lambda$:

$$[T]_{\mathcal{B}_0} = A, \quad [T]_{\mathcal{B}} = \Lambda,$$

with $V = [v_1|v_2]$ as the 'change of basis matrix'.

In the **example**, the basis of $\mathbb{R}^2$ corresponding to the complex eigenvector $(1, 1-i)$ and the matrix of $T$ in that basis (that is, the 'standard form' of $A$) are:

$$\mathcal{B} = \{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix}\}, \quad \Lambda = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}.$$

**Exercise.** For the $2 \times 2$ matrices given below,

(i) Find the eigenvalues (they are complex in both cases);

(ii) Find the (complex) eigenspace for each eigenvalue;

(iii) Compute the 'standard form' $\Lambda$ of the matrix, and the matrix $V$ that takes the given matrix $A$ to standard form (that is, so that $V^{-1}AV = \Lambda$).

$$\begin{bmatrix} -1 & -4 \\ 1 & -1 \end{bmatrix}; \quad \begin{bmatrix} -3 & 5 \\ -2 & 3 \end{bmatrix}.$$