MATH 142- EXAM 5-April 22, 2005

Instructions. Justify answers for full credit. Calculators allowed. Time given: 60 minutes.

1.[12] Determine convergence/divergence for the following series. Justify.

$$\sum_{n=1}^{\infty} \frac{\sin n}{1+n^2}$$

$$\frac{|\sin n|}{1+n^2} \leq \frac{1}{1+n^2} \leq \frac{1}{n^2} \Rightarrow CONV(comparison)$$

$$\sum_{n=1}^{\infty} (-1)^n \frac{n!}{3^n}$$

$$\frac{|a_{n+1}|}{|a_n|} = \frac{n+1}{3} \to \infty \Rightarrow DIV(ratio)$$

$$\sum_{n=1}^{\infty} \frac{(n+1)^2}{n^3(n+2)}$$

$$\frac{(n+1)^2}{n^3(n+2)} \sim \frac{n^2}{n^4} = \frac{1}{n^2} \Rightarrow CONV(limit-comparison)$$

2.[8] Find the number of terms you need to add to approximate each of the infinite sums below with $|error| < 0.01$:

$$(a) \sum_{n=1}^{\infty} \frac{1}{n^{7/2}}; \quad (b) \sum_{n=1}^{\infty} (-1)^n \frac{n}{4^n}$$

$$\int_N^{\infty} \frac{dx}{x^{7/2}} = \frac{2}{5} N^{-5/2} < 10^{-2} \Rightarrow 2.5N^{5/2} > 100, \quad N \geq 5$$

$$\frac{N+1}{4^{N+1}} < 10^{-2}, \quad N = 4 \text{ works}$$

3.[4] Find a representation of the function given below (*choose one!*) as a power series at 0, including the radius of convergence:

$$(a) f(x) = \frac{1}{(x-2)^2} \quad (b) f(x) = \frac{x}{x^2+4}$$

$$[\frac{1}{2-x}]' = \frac{1}{2}[\frac{1}{1-\frac{x}{2}}]' = \frac{1}{2}(\sum_{n=0}^{\infty} \frac{x^n}{2^n})' = \frac{1}{2} \sum_{n=1}^{\infty} \frac{nx^{n-1}}{2^n}, \quad R = 2.$$

$$\frac{x}{x^2+4} = \frac{x}{4}\frac{1}{1+\frac{x^2}{4}} = \frac{x}{4}\sum_{n=0}^{\infty}(-\frac{x^2}{4})^n = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{4^{n+1}} \quad R=2.$$

4.[14] Use a power series to approximate to 5 decimal places the definite integral:

$$\int_0^{0.2} \frac{1}{1+x^5}dx$$

(*4 steps*: (i)[4] expand the integrand as a power series; (ii)[4] compute the definite integral, yielding an alternating series; (iii)[4]use the remainder estimate to compute the number N of terms needed; (iv) [2]compute the partial sum s_N of the series to obtain the approximation.)

$$\int_0^{0.2}\frac{dx}{1+x^5} = \int_0^{0.2}\sum_{n=0}^{\infty}(-1)^n x^{5n}dx = \sum_{n=0}^{\infty}(-1)^n\frac{x^{5n+1}}{5n+1}|_0^{0.2} = \sum_{n=0}^{\infty}(-1)^n\frac{(0.2)^{5n+1}}{5n+1}.$$

$$|r_N| < \frac{(0.2)^{5(N+1)+1}}{5(N+1)+1} = \frac{(0.2)^{5N+6}}{5N+6} < 10^{-5} \text{ (N=1 works)}$$

Approximate value: $0.2 - \frac{0.2^6}{6} = 0.1999893...$