The inverse-square law follows from Kepler's laws.

Kepler's second law (area is swept out by the position vector at a constant rate) is equivalent to the fact that the angular momentum (per unit mass) is a constant of motion:

$$r^2\theta' = l \equiv \text{ const.}$$

In Newton's *Principia* it was demonstrated that Kepler's first law- the planets follow elliptical orbits, with the sun at a focus- *implies* that gravity follows an 'inverse-square law', under the assumption that it is a an attractive central force directed along a line from the planet to the Sun (that is, towards the origin). What follows is Newton's argument in modern notation. Start from the polar equation of an ellipse, with the origin at a focus:

$$r = \frac{a}{1 + e\cos\theta}.$$

Differentiating twice in time and using l to eliminate θ', we find:

$$r' = \frac{ae\sin\theta}{(1+e\cos\theta)^2}\theta' = \frac{le}{a}\sin\theta,$$

$$r'' = \frac{le}{a}(\cos\theta)\theta' = \frac{l^2 e}{ar^2}\cos\theta.$$

Using the relation $e\cos\theta = ar^{-1} - 1$, we can express this in terms of r only:

$$r'' = \frac{l^2}{ar^2}(\frac{a}{r} - 1) = \frac{c^2}{r^3} - \frac{c^2}{ar^2}.$$

Now recall the expression for the radial component of Newton's second law of motion:

$$F_r = ma_r = m(r'' - r(\theta')^2),$$

to obtain (again using l to eliminate θ'):

$$F_r = m(\frac{l^2}{r^3} - \frac{l^2}{ar^2} - r\frac{l^2}{r^4}) = -\frac{ml^2}{a}\frac{1}{r^2},$$

showing that the magnitude of the force follows an inverse-square law.

Problems. The same type of argument (still for a central force, that is, with l as a conserved quantity) allows one to relate different orbit geometries to different power laws for the force. The following examples are taken from

[Tenenbaum-Pollard, p.497]; the problem is to derive the expression for the force from the polar equation for the orbit, exactly as above.

1. Orbit: ellipse with origin at the center.

$$r = \frac{1}{\sqrt{1 - e^2 \cos^2 \theta}} \quad (0 < e < 1), \quad F = -ml^2(1 - e^2)r.$$

([Tenenbaum-Pollard, p.476 ff.] has a discussion of motion under a central force proportional to distance.)

Hint: First derive $r' = -le^2 r \cos\theta \sin\theta$. To eliminate θ in r'', use:

$$e^2 \cos^2 \theta = 1 - \frac{1}{r^2}, \quad e^2 \sin^2 \theta = e^2 - 1 + \frac{1}{r^2}.$$

2. Orbit: spiral $r = e^\theta$; $F = -2ml^2/r^3$.

3. Orbit: lemniscate $r^2 = a^2 \cos 2\theta$. $F = -3ml^2 a^4/r^7$.

4. Orbit: cardioid $r = a(1 + \cos\theta)$. $F = -3ml^2 a/r^4$.

5. Orbit: circle, with the origin at a point on the circumference $r = 2a\cos\theta$. $F = -8a^2 l^2 m/r^5$.