1. THE EXPONENTIAL FUNCTION.

The solution of first-order linear differential equations is based on the exponential, so it is useful to recall its definition and properties. (As a side benefit, we'll review some important facts from one-variable Calculus).

The exponential function is *defined* as the inverse of natural logarithm. In turn, the natural logarithm is defined as a definite integral:

$$L(x) = \int_1^x \frac{dt}{t}, \quad x > 0.$$

$L(x)$ is continuous for $x > 0$ (in fact, differentiable with $L'(x) = 1/x$, by the fundamental theorem of calculus), positive for $x > 1$, negative for $0 < x < 1$, strictly increasing. From this definition we can prove the basic property:

$$L(xy) = L(x) + L(y) \quad \text{for } x, y > 0.$$

(proof seen in class.)

Recall the *inverse function theorem* of Calculus:

Theorem.(a) If $f(x)$ is strictly increasing and continuous on an interval (a, b), *onto* an interval (c, d), then the inverse function $g(y)$ is defined, strictly increasing and continuous on (c, d). By definition of 'inverse function':

$$g(f(x)) = x \text{ for all } x \in (a, b); \quad f(g(y)) = y \text{ for all } y \in (c, d).$$

(b) If, in addition, f is differentiable with $f'(x) \neq 0$ on (a, b), then g is differentiable on (c, d) with $g'(y) = \frac{1}{f'(g(y))}$.

Note that the domain of the inverse function is the interval (c, d) covered by all values of f on (a, b). What is the interval covered by $L(x)$ on $[1, \infty)$? If it were a bounded interval $[0, A]$, this would imply the total area under the graph of $1/t$ for $0 < t < \infty$ would be finite (equal to A), which we know is not true (use the integral test and divergence of the harmonic series). Similarly $L(x)$ for x in the interval $(0, 1)$ must cover the whole negative real line; otherwise the integral from 0 to 1 of $1/t$ would be finite, which is not true. We conclude $L(x)$ is *one-to-one onto* ('bijective') from $(0, \infty)$ to the whole real line. Thus the domain of its inverse is all of $\mathbb{R}$.

Definition. The exponential function $E : \mathbb{R} \to \mathbb{R}_+$ is the inverse function of the natural logarithm $L(x)$.

Now, given two real numbers y_1, y_2, there are unique positive real numbers x_1, x_2 so that $y_1 = L(x_1)$ and $y_2 = L(x_2)$. Then:

$$E(y_1 + y_2) = E(L(x_1) + L(x_2)) = E(L(x_1 x_2)) = x_1 x_2 = E(y_1)E(y_2),$$

using the basic property of $L(x)$ and the definition of 'inverse function'. That is, E transforms sums into products. In particular, for any natural number n, we find easily that:

$$E(n) = E(1)^n,$$

and similarly for fractional powers p/q:

$$E(p/q) = E(1)^{p/q}$$

(since $[E(p/q)]^q = E(\frac{p}{q}q) = E(p) = E(1)^p$.) Thus, at least for rational numbers x, the function $E(x)$ coincides with A^x, where $A = E(1)$. *Question:* what is the number A?

To answer this, we have to look at values of $L(x)$ for x close to 1. We start by making a change of variable (for $x > 1$):

$$L(x) = \int_0^{x-1} \frac{dt}{1+t}.$$

Then recall the expansion, convergent for $|t| < 1$ (geometric series):

$$\frac{1}{1+t} = 1 - t + t^2 - t^3 + \ldots$$

Integrating:

$$L(1+a) = \int_0^a (1 - t + t^2 + \ldots)dt = a - \frac{a^2}{2} + \frac{a^3}{3} + \ldots$$

(convergent for $0 < a < 1$). Now use this for $a = 1/n$, $n > 1$ (and multiply by n):

$$nL(1 + \frac{1}{n}) = 1 - \frac{1}{2n} + \frac{1}{3n^2} + \ldots$$

We see that:

$$\lim_{n\to\infty} nL(1 + \frac{1}{n}) = 1,$$

or:

$$L[(1 + \frac{1}{n})^n] \to 1 \text{ as } n \to \infty.$$

Now apply the exponential function E to both sides of this limit (this is legitimate, since E is a continuous function.). We obtain:

$$(1+\frac{1}{n})^n \to E(1) \text{ as } n \to \infty.$$

Note: this 18th. century argument (L.Euler) is not completely rigorous by late 20th. century standards. By the end of the 21st., who knows?

With this formula we can compute $E(1)$ to an arbitrary degree of precision, and find it equals the number usually denoted by e:

$$E(1) = 2.718281828... = e.$$

This justifies the conventional *notation* for the exponential function:

$$E(x) = e^x.$$

(Keep in mind this is just a notation-giving something a different name-and contains no new mathematical information beyond the definition given above.)

Now consider derivatives. From (ii) in the inverse function theorem, we find, for any $y \in \mathbb{R}$:

$$E'(y) = \frac{1}{L'(E(y))} = \frac{1}{1/E(y)} = E(y).$$

That is, the exponential function equals its own derivative! Another way of saying this is that E is a solution of the 'first order differential equation':

$$f' = f.$$

More generally, for any real number α, we have from the chain rule:

$$E'(\alpha x) = \alpha \frac{d}{dy} E(y)|_{y=\alpha x} = \alpha E(\alpha x).$$

That is, using now the conventional notation $e^{\alpha x}$ for $E(\alpha x)$, we find that $f(x) = e^{\alpha x}$ is a solution of the DE:

$$f' = \alpha f.$$

A basic question is: *are there any other solutions?* The answer is yes, but any other solution differs from this one simply by multiplication by a constant:

Proposition. If $g(x)$ is any differentiable function satisfying $g' = \alpha g$ (where $\alpha \in \mathbb{R}$ is a constant) and $f(x) = e^{\alpha x}$, then $g(x) = Cf(x)$ for some constant $C \in \mathbb{R}$.

Proof. Since $f(x) > 0$, we apply the 'quotient rule' to the function $g(x)/f(x)$:

$$(\frac{g}{f})' = \frac{g'f - gf'}{g^2} = \frac{\alpha gf - g\alpha f}{g^2} = 0,$$

so g/f is a constant C.

This proposition says that the 'general solution' to the DE $f' = \alpha f$ is given by:

$$f(x) = Ce^{\alpha x},$$

where C is an arbitrary constant. Substituting $x = 0$ in this relation, we find $C = f(0)$, or:

$$f(x) = f(0)e^{\alpha x}.$$

The 'physical interpretation' of this conclusion is the following: if $f(t)$ denotes a positive quantity (mass of a sample, population, etc.) which changes in time, $f'(t)/f(t)$ is the 'instantaneous relative growth rate': the rate of change at time t divided by the value of f at time t. Saying $f' = \alpha f$ means the 'relative growth rate' is constant in time, equal to α. The conclusion is that, if $\alpha > 0$, the value of f increases exponentially fast while, if $\alpha < 0$, f *decreases* to zero exponentially fast. These are the only possible 'long term behaviors' under a 'constant relative growth rate'.

EXERCISES

1. Let $f(x) = e^{\alpha x} + L$ (for constants α, L). What is the 'first-order differential equation' solved by f? (The variable x should not appear in the equation). What is the 'general solution' of this DE? (*Hint:* consider the equation solved by $g(x) = f(x) - L$).

(*Interpretation:* The DE has a constant 'source term' (a contribution to the rate of change independent of the value of f). If $\alpha < 0$, the solution 'stabilizes' to L for large t (decreases to L if $f(0) > L$, increases to L if $f(0) < L$), where $L > 0$ if the source term is positive, $L < 0$ if the source term is negative. Draw some graphs!)

2. The 'hyperbolic sine' and 'hyperbolic cosine' functions are defined (respectively) by:

$$\sinh(x) = \frac{1}{2}(e^x - e^{-x}), \quad \cosh(x) = \frac{1}{2}(e^x + e^{-x}).$$

(i) Show that both functions are solutions of the 'second-order linear DE': $f'' = f$.

(ii) Find two solutions of the DE $f'' = \alpha^2 f$, where $\alpha \in \mathbb{R}$. (*Hint:* replace x by αx in (i)).

3. The 'hyperbolic tangent' and 'hyperbolic cotangent' are defined by:

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)}, \quad \coth(x) = \frac{\cosh(x)}{\sinh(x)}.$$

Draw their graphs. Show that both are solutions of the 'non-linear first-order DE':

$$f' + f^2 - 1 = 0.$$

What is the non-linear first-order DE solved by $\tanh(\alpha x)$, $\coth(\alpha x)$?

4. The functions $\tan(\alpha x)$ and $\cot(\alpha x)$ (the usual tangent and cotangent, evaluated at αx) solve non-linear first-order DEs similar to that stated in problem 3. Find them.

Remark: The family of DE: $f' \pm f^2 + k = 0$, where $k \in \mathbb{R}$, is known as 'Ricatti equations'. It is one of the very few non-linear DE admitting 'explicit closed-form solutions'.