MATHEMATICS 251-LINEAR ALGEBRA- FINAL EXAM, 12/15/2005

Instructions. 2h exam- closed book, closed notes. Calculators allowed (but not needed.) No partial credit for answers given without justification, even if correct- show all work! The exam consists of **five** problems (10pts per item.)

1. The matrix

$$A = \begin{bmatrix} 0 & 5 & 0 \\ -2 & 6 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

has $3+i$ as an eigenvalue, with eigenvector $\begin{bmatrix} 2+i \\ 1+i \end{bmatrix}$. (In addition, note $Ae_3 = 5e_3$.) Find (i) the 'standard form' Λ of A, and (ii) an invertible matrix P so that $P^{-1}AP = \Lambda$.

2. The characteristic polynomial of the matrix:

$$B = \begin{bmatrix} 1 & 12 & 0 \\ -1 & 8 & 0 \\ 1 & -3 & 4 \end{bmatrix}$$

is $p(\lambda) = (\lambda - 4)^2(\lambda - 5)$.

(i) Find bases for the eigenspaces of B. (ii) Is B diagonalizable? Justify.

3. The 3×3 matrix P projects vectors in $\mathbb{R}^3$ orthogonally onto the plane $x + 2y + 3z = 0$.

(i) Find the eigenvalues and eigenspaces of P;

(ii) Find $P^{1357}\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$.

4. A 2×2 symmetric matrix C has eigenvalues $1/5$ and 2. The eigenspace $E(2)$ is spanned by the vector $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$. Let $v = a\begin{bmatrix} 2 \\ 3 \end{bmatrix} + b\begin{bmatrix} -3 \\ 2 \end{bmatrix}$, where a, b are arbitrary real numbers.

(i) Find $C^n v$, for arbitrary $n \geq 0$;

(ii) Describe the behavior of $C^n v$ as $n \to \infty$ (sketch the eigenspaces of C and the vector $C^n v$, for large n.)

*(**Problem 5** is on the back of the page.)*

5. (i) Find the maximum and minimum values of the quadratic form in $\mathbb{R}^2$:

$$Q(x, y) = 2x^2 + 4xy - y^2$$

on the circle $x^2 + y^2 = 1$, and the points on the circle where the max/min values are attained.

(ii) Find a rotation of coordinates that eliminates the xy term in the equation of the conic $2x^2 + 4xy - y^2 = -8$; write the equation of the conic in the new coordinates and identify it (as an ellipse, hyperbola or parabola).

(*Hint:* Recall the new coordinates $x'y'$ will be related to the original coordinates xy by:

$$\begin{bmatrix} x \\ y \end{bmatrix} = R \begin{bmatrix} x' \\ y' \end{bmatrix},$$

where R is a 2×2 orthogonal matrix that diagonalizes the 2×2 symmetric matrix associated with Q.)