LINEAR ALGEBRA- REVIEW PROBLEMS (Eigenvalues, powers of operators)

1. A is 2×2 symmetric, with eigenvalues $1/2$ and 3. $E(3)$ is spanned by $(1,2)$. Let $v = x(1,2)+y(-2,1)$. (i) Find $A^n v$ ($n\geq 1$ arbitrary); (ii) What happens to $A^n v$ as $n\to\infty$?

Solution: Since A is symmetric, the two eigenspaces are orthogonal, so $E(1/2)$ is spanned by (say) $(-2,1)$. Thus $A^n(1,2)=3^n(1,2)$ and $A^n(-2,1)=(1/2^n)(-2,1)$. By linearity, $A^n v = 3^n x(1,2)+(1/2^n)y(-2,1)$. The component along $(1,2)$ tends to infinity, while the component along $(-2,1)$ tends to zero. Thus $A^n v$ approaches the eigenspace $E(3)$ as $n\to\infty$ (and its length tends to infinity).

2. $P:\mathbb{R}^3\to\mathbb{R}^3$ is the matrix of orthogonal projection onto the plane $x+2y+z=0$. (i) Find the eigenvalues and eigenspaces of P. (ii) Compute the limit $\lim_{n\to\infty} P^n(1,1,1)$.

Solution. Let E be the given plane, $E^\perp$ the orthogonal line. $Pv=v$ if $v\in E$, $Pv=0$ if $v\in E^\perp$, and E and $E^\perp$ together span $\mathbb{R}^3$. Hence the eigenvalues are 1 (with eigenspace E) and 0 (with eigenspace $E^\perp$.).(In particular, P is diagonalizable.) The projections of $v=(1,1,1)$ on $E^\perp$ and E are (with $u=\frac{1}{\sqrt{6}}(1,2,1)$, the unit normal vector to E)::

$$P^\perp v = \langle v,u\rangle u = \frac{2}{3}(1,2,1), \quad Pv = v - P^\perp v = (1/3,-1/3,1/3),$$

and applying P again to Pv won't change it, so $P^n v=(1/3,-1/3,1/3)$ for all $n\geq 1$.

3.$T:\mathbb{R}^2\to\mathbb{R}^2$ expands every vector in the plane by a factor of 2, while rotating it by an angle $\pi/4$ (counterclockwise). (i) What are the eigenvalues of T? (ii) Show that T^4 fixes every line through the origin.

Solution.

$$T = 2R_{\pi/4} = \begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{bmatrix},$$

so the eigenvalues are $\sqrt{2}\pm i\sqrt{2}$. For the 4th. power T^4, we have $T^4 = 2^4 R_\pi = -2^4 I$. Since it is a multiple of the identity, T^4 fixes every line through 0.

4.Let $R:\mathbb{R}^3\to\mathbb{R}^3$ be the rotation matrix with axis spanned by $(1,1,2)$, by an angle $\pi/3$ (looking down the axis). (i) What are the eigenvalues of R? (ii)What is the 'standard form' matrix of R?

Solution. Vectors v on the axis are fixed by R ($Rv = v$), so 1 is an eigenvector with one-dimensional eigenspace spanned by $(1, 1, 2)$. On the orthogonal plane, R is the rotation matrix:

$$R_{\Pi/3} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix},$$

with eigenvalues $(1/2) \pm i\sqrt{3}/2$, which are also (complex) eigenvalues of R. The matrix of R in an appropriate basis is given by a rotation block, followed by a 1 on the diagonal:

$$\Lambda = \begin{bmatrix} 1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

5. Let $S : \mathbb{R}^3 \to \mathbb{R}^3$ be *reflection* on the plane $x + 2y + z - 0$. (i) What are the eigenvalues and eigenspaces of S? (ii)Find $S^{2n}(1, 1, 0)$ and $S^{2n+1}(1, 1, 0)$, for each $n \geq 1$.

Solution. S fixes vectors on the given plane (call it E), meaning $Sv = v$, and 'flips' vectors on the orthogonal line $E^\perp$ (meaning $Sv = -v$). Hence the eigenvalues are 1 (with eigenspace E) and -1 (with eigenspace $E^\perp$.) To find $S(1, 1, 0)$, we decompose $v = (1, 1, 0)$ into components on $E^\perp$ (spanned by the unit vector $u = (1/\sqrt{6}))(1, 2, 1)$ and on E:

$$P^\perp v = \langle v, u\rangle u = (1/2)(1, 2, 1), \quad Pv = v - P^\perp v = (1/2, 0, -1/2),$$

then compute the action of S:

$$Sv = Pv - P^\perp v = (1/2, 0, -1/2) - (1/2, 1, 1/2) = (0, -1, -1).$$

Reflecting twice (or any even number of times) doesn't move the vector at all (so $S^{2n}v = v$ for all $n \geq 1$), while reflecting an odd number of times is the same as reflecting once, so $S^{2n+1}v = Sv = (0, -1, -1)$ for all $n \geq 1$.

Remark. Here we could have used the standard formula for reflections derived in class, $Sv = v - 2\langle v, u\rangle u = (1, 1, 0) - (1, 2, 1) = (0, -1, -1)$, to find Sv.