MATH 251- EXAM 1- September 16, 2005

1. For the matrix A and vector b given below:

(i) Find the general solution (in vector parametric form) of the system $Ax = b$, $x = (x_1, x_2, x_3, x_4)$;

(ii) Find the general solution (in vector parametric form) of the homogeneous system $Ax = 0$ (i.e., the nullspace of A.)

$$b = (4, -1, 3, -5)$$

$$A = \begin{bmatrix} 1 & 2 & -3 & 1 \\ -2 & 1 & 2 & 1 \\ -1 & 3 & -1 & 2 \\ 4 & -7 & 0 & 5 \end{bmatrix}.$$

Row reduction of the augmented matrix $[A||b]$ *yields:*

$$\begin{bmatrix} 1 & 2 & -3 & 1 & 4 \\ 0 & 5 & -4 & 3 & 7 \\ 0 & 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Solving the resulting system by back-substitution gives the general solution in parametric form:

$$x = (6/5, 7/5, 0, 0) + t(7/5, 4/5, 1, 0), \quad t \in \mathbb{R}.$$

In particular, the general solution of the homogeneous system is:

$$x_h = t(7, 4, 5, 0), \quad t \in \mathbb{R}.$$

2. Determine whether $b = (1, 2, 3, -1)$ is in the column space of A and if so express b as a linear combination of the columns of A.

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 1 & 3 & 2 & 2 \\ 0 & 1 & 2 & 2 \end{bmatrix}$$

Row-reducing the augmented matrix $[A||b]$*, we find:*

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 1 \\ 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Solving the reduced system by back-substitution yields the general solution x= (-6+2t, 5-2t, t, -3). *Setting t=0 we find:*

$$(1, 2, 3, -1) = -6(1, 0, 1, 0) + 5(2, 1, 3, 1) - 3(1, 1, 2, 2).$$

(Other answers possible, depending on the choice of t)

3. Find parametric equations (in vector form) for the line of intersection of the planes $2x + 3y - z = 1$ and $x + y - 2z = 2$.

Again by Gaussian elimination, we find that the system given by these two equations has as general solution:

$$(x, y, z) = (5 + 5t, -3t - 3, t) = (5, -3, 0) + t(5, -3, 1).$$

This is the desired parametric equation for the line of intersection.

4. Let k be an arbitrary non-zero real number. Find the inverse of the 4X4 matrix (include all the steps for credit):

$$A = \begin{bmatrix} k & 0 & 0 & 0 \\ 1 & k & 0 & 0 \\ 0 & 1 & k & 0 \\ 0 & 0 & 1 & k \end{bmatrix}$$

Answer:

$$A^{-1} = \frac{1}{k^2} \begin{bmatrix} k & 0 & 0 & 0 \\ -1 & k & 0 & 0 \\ 1/k & -1 & k & 0 \\ -1/k^2 & 1/k & -1 & k \end{bmatrix}.$$

5. Find the conditions the b_i must satisfy for the system to be consistent:

$$\begin{cases} x_1 & -2x_2 & +5x_3 & = b_1 \\ 4x_1 & -5x_2 & +8x_3 & = b_2 \\ -3x_1 & +3x_2 & -3x_3 & = b_3 \end{cases}$$

Row-reducing $[A\|b]$ for a general vector b, we find:

$$\begin{bmatrix} 1 & -2 & 5 & b_1 \\ 0 & 3 & -12 & b_2 - 4b_1 \\ 0 & 0 & 0 & -b_1 + b_2 + b_3 \end{bmatrix}.$$

Thus the condition is: $b_1 = b_2 + b_3$ (Incidentally, this is a *defining equation* for $Ran(A)$.)