MATH 251- EXAM 2- October 21, 2005

1. Consider the matrix A given below, and the result A' of applying row reduction to A.

$$A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 3 & 4 & 2 & 2 \\ 2 & 2 & 2 & 3 \end{bmatrix} \longrightarrow A' = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & -2 & 2 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(i)[10] Find the rank of A, and the dimensions of the spaces $Row(A)$, $N(A)$, $Col(A)$, $N(A^T)$. Justify.

Since A' has two non-zero rows, rank(A)=2, which is the same as the dimensions of $Col(A)$ and $Row(A)$; then $N(A)$ has dimension $4-2=2$ and $N(A^T)$ has dimension $3-2=1$.

(ii)[10] Write down parametric equations for the spaces $Row(A)$ and $Col(A)$.

T*he first two columns (resp. rows) of A (resp. A') give bases for $Col(A)$ (resp. $Row(A)$), hence:*

$$Row(A) = \{s(1,2,0,-1) + t(0,-2,2,5); s,t \in \mathbb{R}\};$$

$$Col(A) = \{u(1,3,2) + v(2,4,2); u,v \in \mathbb{R}\}.$$

(iii)[20] Write down a system of independent defining equations for the space $N(A)$.

Since $N(A) = N(A')$, by definition of nullspace we get from the rows of A' the equations:

$$x_1 + 2x_2 - x_4 = 0, \quad -2x_2 + 2x_3 + 5x_4 = 0.$$

2.[20] Let W be the subspace of $\mathbb{R}^5$ defined as the solution space of the homogeneous system:

$$\begin{cases} x_1 & +x_2 & +x_3 & +x_4 & +2x_5 & = 0 \\ 2x_1 & -x_2 & -x_3 & & & = 0 \end{cases}$$

Find a basis for the orthogonal complement $W^\perp$ of W.

The two equations are linearly independent, so dim $W = 5-2 = 3$ and dim $W^\perp = 5-3 = 2$. W is the nullspace of the 2×5 matrix defining the system, so $W^\perp$ is the row space, with a basis given by the coefficient vectors of the two equations:

$$\mathcal{B} = \{(1,1,1,1,2), (2,-1,-1,0,0)\}.$$

3.[20] Find a defining equation for the subspace of $\mathbb{R}^4$ spanned by:

$$\{(1,0,-1,1),(0,1,1,2),(0,-1,1,0)\}$$

(*Hint:* These vectors are linearly independent; the general equation for a three-dimensional subspace of $\mathbb{R}^4$ has the form $Ax + By + Cz + Dw = 0$, where you may assume $D = -1$).

As suggested, we write the equation of the subspace as:

$$Ax + By + Cz = w;$$

using the coordinates of the given points yields the system:

$$A - C = 1, \quad B + C = 2, \quad -B + C = 0,$$

which is easily solved for A, B and C, giving the equation:

$$2x + y + z - w = 0.$$

4.Let $\mathcal{B}$ be the basis of $\mathbb{R}^2$:

$$\mathcal{B} = \{v_1, v_2\}, \quad v_1 = (1,2), v_2 = (1,3).$$

(i)[10]If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by:

$$Tv_1 = (1/2)v_1 + v_2, \quad Tv_2 = 2v_1 - v_2,$$

find $[T]_{\mathcal{B}}$. *From the definition of $[T]_{\mathcal{B}}$, we get immediately:*

$$[T]_{\mathcal{B}} = \begin{bmatrix} 1/2 & 2 \\ 1 & -1 \end{bmatrix}.$$

(ii)[10] For the same transformation T, find $T(1,0)$ and $T(0,1)$.
Since $(1,0) = 3v_1 - v_2$ *and* $(0,1) = v_2 - v_1$,

$$T(1,0) = 3Tv_1 - Tv_2 = (3/2)v_1 + 3v_2 - (v_1 + 2v_2) = (1/2)v_1 + v_2 = (3/2, 4)$$

$$T(0,1) = Tv_2 - Tv_1 = 2v_1 - v_2 - ((1/2)v_1 + v_2) = (3/2)v_1 - 2v_2 = (-1/2, -3)$$