Eudoxus' METHOD OF EXHAUSTION, as used by ARCHIMEDES

Archimedes, MEASUREMENT OF A CIRCLE

Proposition 1. *The area of any circle is equal to a right-angled triangle in which one of the sides about the right angle is equal to the radius, the other to the circumference, of the circle.*

In symbols, if r is the radius and C is the circumference, we wish to show the area of the circle equals $(1/2)rC$. Let T denote this number.

1. Starting with an inscribed square and by successive bisection, exhaust the circle from the inside by a sequence of regular polygons P_n with 2^n sides, $n \geq 1$.

Observe that, at each subdivision, the area of the circle still to be absorbed is less than half the area left to be absorbed at the previous stage:

$$area(circle) - area(P_{n+1}) < \frac{1}{2}(area(circle) - area(P_n)).$$

For example, in going from the square $ABCD$ to the octagon, the area of the triangle AEB is more than half the area of the circular segment AEB, since the rectangle $AFGB$, which has area twice that of the triangle, encloses the circular segment. Thus the sum of the areas of the circular sectors defined by EB and EA is less than half the area of the sector AEB, as we claimed. Thus, by *Euclid X.1*, these 'errors' $area(circle) - area(P_n)$ will be smaller than any preassigned number, if n is large enough.

2. On the other hand, by adding triangles we see that $area(P_n) = (1/2)h_nC_n$, where h_n is the distance from the center of the circle to a side of P_n, and C_n is its perimeter. Since $h_n < r$ and $C_n < C$, it is clear that $area(P_n) < T$.

3. But then the area of the circle cannot be greater than T. If it were, consider the inequalities: $area(P_n) < T < area(circle)$. As observed in 1., the gap between $area(P_n)$ and $area(circle)$ can be made arbitrarily small, so at some point $area(P_n)$ would have to be greater than T, which as seen in 2. is not possible.

4. We now repeat the argument from the outside, beginning with a circumscribed square Q_1 and obtaining, by successive subdivision, a sequence Q_n of circumscribed polygons with 2^n sides. By adding the areas of the triangles (of height r) making up Q_n, we see that $area(Q_n) = (1/2)rL_n$, where L_n is the perimeter of Q_n. Since L_n is greater than C, $area(Q_n)$ is certainly greater than T.

5. Observe that, in moving from the circumscribed square Q_1 to the octagon Q_2, we throw away more than half of the excess area $area(Q_1) - area(circle)$. Indeed, consider the triangles TFG and TEH; here FG is a side of the octagon, touching the circle at A, while TE and TH are contained in sides of the square. Since the triangle AGT has a right angle at A, TG is greater than AG; AG being equal to TH, this implies TG is greater than TH; so twice the area of the triangle TFG is greater than the area of the triangle TEH, which *contains* the excess region $TEAH$. Thus the triangle TEH, the part of the excess region about to be discarded, is greater in area than half the excess region; of course, this holds true for all n:

$$area(Q_{n+1}) - area(circle) < (1/2)(area(Q_n) - area(circle));$$

so again it follows from *Euclid X.1* that this 'error' can be made arbitrarily small, for n large enough.

6. But now we see that we can't have $area(circle) < T$, either: if this were true, consider the double inequality $area(circle) < T < area(Q_n)$. As seen in 5., the gap between $area(circle)$ and $area(Q_n)$ can be made as small as desired; so at some point $area(Q_n)$ has to drop below T, which as seen in 4. is not possible. So the only possibility left is $area(circle) = T$.

Archimedes' QUADRATURE OF A PARABOLA

Proposition 23. *The area of a segment of a parabola bounded by a chord Qq is equal to four-thirds the triangle with the same base Qq and height as the parabolic segment.*

The *vertex* of a parabolic segment is the unique point P on the parabola whose distance to the chord Qq is maximal; equivalently, the tangent to the parabola at the vertex P of a segment is parallel to the chord Qq defining the segment. A *diameter* of the parabolic segment is any line through a point on the parabola and parallel to the axis of the parabola; the *distance* from P to Qq (in general different from the length of PV) is the 'height' of the parabolic segment.

The proposition states the area of the segment is (4/3) of the area of the triangle QPq. Let T be this number.

Two facts about parabolas are used in the proof.

Property 1: the diameter through the vertex P bisects the chord Qq (at the point V);

Property 2: consider any other chord Q_1q_1, parallel to Qq, and hence defining a parabolic segment with the same vertex P; let PV_1 be the diameter through P. The ratio $(Q_1V_1)^2/PV_1$ is constant over all such chords.

(The reader may wish to think first of a 'right' parabolic segment, easier on Cartesian eyes; then the two properties are evident.)

The proof proceeds by exhausting the area from the inside, by inscribed polygons. Instead of also approximating from the outside, as for the circle, Archimedes relies on the fact that the approximating areas are partial sums of a geometric series with ratio 1/4.

The first inscribed approximant is the triangle $P_1 = QPq$ itself.

1. The excess area $area(segment) - area(P_1)$ is less than half the area of the segment. This follows since the parallelogram defined by Qq, the tangent at P, and the diameters through Q and q has area twice the area of the triangle, and contains the whole segment; so the area of the triangle is greater than half the area of the segment.

2. Let R be the midpoint of QV, so the diameter through R defines a new parabolic segment QQ_1P, with vertex Q_1; let V_1 be the midpoint of the segment QP. We claim the area of the triangle QQ_1P is one-quarter of the area of the triangle QPV. It suffices to show that $Q_1V_1 = \frac{1}{2}V_1R$; for this shows the area of the triangle PQ_1V_1 is half the area of the triangle PV_1R, while the area of QV_1Q is half that of QV_1R; adding these two, we get:

$$area(\Delta QQ_1P) = \frac{1}{2}area(\Delta QPR) = \frac{1}{4}area(\Delta QPV).$$

Denote by X_1, W_1 the footpoints on the diameter PV of line segments drawn from Q_1 and V_1 (resp.), parallel to Qq. Since Q_1X_1 is a half-chord parallel to Qq, Property 2 implies $PX_1/(Q_1X_1)^2 = PV/(QV)^2$, hence:

$$PX_1 = (\frac{QX_1}{QV})^2 PV = \frac{1}{4}PV,$$

while $W_1V = V_1R = (1/2)PV$, by similarity of triangles. Hence:

$$Q_1V_1 = X_1W_1 = PV - PX_1 - W_1V = PV - \frac{1}{4}PV - \frac{1}{2}PV = \frac{1}{4}PV = \frac{1}{2}V_1R,$$

as claimed.

3. We see that, repeating this process, we get a sequence of inscribed polygons P_n so that: (i) at each stage, the area added is (1/4) of the preceding area:

$$area(P_{n+1}) - area(P_n) = \frac{1}{4}area(P_n);$$

thus $area(P_n)$ is the sequence of partial sums of a geometric sequence with first term the area of the triangle QPq, ratio 1/4; (ii) at each stage, the area left to absorb in the parabolic segment is less than (1/2) the combined area of the triangles just added:

$$area(segment - P_n) < \frac{1}{2} area(P_n - P_{n-1}).$$

(This follows from repeated use of 1.). In particular, by Euclid X.1, this excess area is smaller than any preassigned magnitude, if n is taken large enough.

4. Geometric sequences and their sums are addressed in Euclid's *Elements*, so Archimedes knew that if A_n are terms of a geometric series with ratio (1/4), then for each $n \geq 1$:

$$A_1 + A_2 + \ldots + A_n + \frac{1}{3} A_n = \frac{4}{3} A_1$$

(*exercise.*) In particular, in the present case this says:

$$area(P_n) + \frac{1}{3} area(P_n - P_{n-1}) = \frac{4}{3} area(P_1) = T,$$

for each $n \geq 2$.

5. Now we see that the area of the segment cannot be greater than T. If it were, consider the double inequality $area(P_n) < T < area(segment)$, where the first one follows from 4. We saw in 3. that the gap between $area(P_n)$ and $area(segment)$ can be made as small as desired, by taking n large enough; thus for some n we must have $area(P_n) > T$, contradiction.

6. On the other hand, since $area(P_n) - area(P_{n-1})$ is a geometric sequence with ratio 1/4, it too can be made as small as desired, so (from 4.) the gap $T - area(P_n)$ becomes as small as desired, as n increases. But then we can't have $area(segment) < T$, for this would imply $area(P_n) < area(segment) < T$, while the fact that the gap between $area(P_n)$ and T is narrowing forces $area(P_n)$ to be eventually greater than the $area(segment)$, contradiction. So we must have $area(segment) = T$, as claimed in the Proposition.