CONVERGENCE OF FOURIER SERIES

1. Periodic Fourier series. The Fourier expansion of a 2π-periodic function f is:

$$f(x) \sim \frac{a_0}{2} + \sum_{n\geq 1} a_n \cos(nx) + b_n \sin(nx),$$

with coefficients given by:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx \quad (n \geq 0), \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx \quad (n \geq 1).$$

The symbol $\sim$ above reminds us that this is a merely formal expansion- the formal series of functions on the right will diverge in many cases, and one needs to understand in what sense it approximates f.

The 'Fourier sine and cosine series', that is, the representations of f as infinite series of eigenfunctions in $[0, \pi]$ with Dirichlet (resp, Neumann) boundary conditions is a special case of this. It is very easy to see that a_n vanishes if f is an odd function, while b_n vanishes if f is even. Thus the analysis of Fourier cosine (resp. sine) series of a function f in $[0, \pi]$ is equivalent to the analysis of the extension of f, first to $[-\pi, \pi]$ as an even (resp. odd) function, then to all of $\mathbb{R}$ with period 2π.

This remark also helps us choose a natural space of periodic functions to work with, namely, piecewise continuous functions. Much as one might prefer to work with continuous ones, the extension (as above) to $\mathbb{R}$ of a continuous function in $[0, \pi]$ is, in general, only piecewise continuous (recall this means f is continuous at all but a finite number of points in $[0, \pi]$, and at discontinuities the one-sided limits exist and are finite; in particular, f is *bounded*.) We adopt the notation C_{pw} for 2π-periodic, piecewise continuous functions *with piecewise-continuous first-derivatives.* More generally, we say $f \in C^k_{per}$ if it is 2π-periodic, with continuous derivatives up to order k and $f^{(k+1)}$ is piecewise continuous; $f \in C_{per}$ means f is 2π-periodic, continuous, and f' is piecewise continuous.

2. Types of convergence. We are interested in the convergence of the sequence of partial sums:

$$s_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos nx + b_n \sin nx.$$

Three types of convergence will be considered:

1. $f_n \to f$ *pointwise* on a set I if $f_n(x) \to f(x)$ for all $x \in I$;

2. $f_n \to f$ *uniformly* on a set I if:

$$\sup_{x \in I} |f_n(x) - f(x)| \to 0.$$

('sup' can be thought of as 'maximum value')

3. $f_n \to f$ *in L^2 norm* if

$$\int_{-\pi}^{\pi} (f_n - f)^2 \to 0.$$

Pointwise convergence is fairly weak: for example, the pointwise limit of continuous functions is in general not continuous (example: $f_n(x) = x^n$ in $[0, 1]$). Uniform convergence is the most natural concept (f is continuous if the f_n are and convergence is uniform), but for this very reason is too much to hope for in many cases. L^2 convergence corresponds to 'approximation in an average sense', but it is not easy to see what it means. For one thing, if $f_n \to f$ in L^2, you can change the f_n and f on a set of measure 0 (for example, on any finite set), and the statement will still be true: integrals are insensitive to what happens in sets of measure 0, so in particular if a sequence of functions has a limit in the L^2 sense, the limit is never unique! Horrible! You would have to declare two functions to be 'equal' if they differ on a set of measure 0, but that opens a whole new can of worms. A closely related concept, but easier to grasp, is *'convergence in L^1 norm'*:

$$\int_{-\pi}^{\pi} |f_n - f| \to 0.$$

Geometrically, this means the area of the region bounded by the graphs of f_n and f (say, in $[-\pi, \pi]$) tends to zero. (And I mean the *geometric* area, not the 'signed area' of Calculus; that's the effect of the absolute value in the definition.) Unfortunately, this interpretation does not hold for L^2 convergence.

There are important logical relationships between these notions of convergence. For instance, uniform convergence easily implies pointwise convergence, L^2 convergence and L^1 convergence (this is really easy! *Check it!*). Also, consider the chain of inequalities:

$$\int_a^b |f_n - f| \le |a - b|^{1/2} \left(\int_a^b (f_n - f)^2\right)^{1/2} \le |a - b|^{1/2} \left(2M \int_a^b |f_n - f|\right)^{1/2}.$$

The first one holds for any functions f_n, f in (say) $C_{pw}[a,b]$- it is the 'Cauchy-Schwarz inequality' for integrals. It shows immediately that *convergence in L^2 norm implies convergence in L^1 norm* (on any bounded interval). The second one holds under the assumption that f and f_n are bounded by M (just use $(f_n(x) - f(x))^2 \leq 2M|f_n(x) - f(x)|$ if $|f_n| \leq M, |f| \leq M$). So it shows that if the sequence f_n is *uniformly bounded* (i.e., all $|f_n|$ bounded by a constant M independent of n) and converges to f in the L^1 sense, then it also converges in the L^2 sense. (Note that, in this case $|f| \leq M$ also holds.)

To get a better grasp of the distinction between these different notions of convergence, there is no substitute for the following exercise.

Exercise 1. Consider the sequence of functions in $[0,1]$:

$$f_n(x) = \frac{M_n}{h_n^2} x^2 \text{ on } [0, h_n], \quad f_n(x) = \frac{M_n}{h_n^2}(2h_n - x)^2 \text{ on } [h_n, 2h_n],$$

and $f_n(x) = 0$ elsewhere in $[0,1]$. We *assume* $h_n \to 0$. M_n is an arbitrary sequence of positive numbers.

(i) Sketch the graphs of the f_n. (ii) Show that $f_n \to 0$ pointwise, regardless of how the M_n are chosen; (iii) Show that $f_n \to 0$ uniformly if $M_n \to 0$, but not otherwise; (iv) Compute $\int_0^1 f_n$ and $\int_0^1 f_n^2$ (use the symmetry!); (v) Show that if the sequence (M_n) is bounded, then $f_n \to 0$ in L^2 norm (hence in L^1 norm). (vi) Show that, by suitably choosing M_n, one can have examples with: (a) $f_n \to 0$ in L^1 norm, but not in L^2 norm; (b) f_n not converging to zero in L^1 norm.

Part (vi)(a) shows that L^2 convergence to zero means a bit more than 'areas under the graph converge to zero'. It is interesting that I had to use a quadratic function to get this example- it doesn't work with linear 'tents'! Also, the examples of part (vi) cannot work if M_n is bounded, as the following interesting proposition shows:

Proposition. If $f_n \to f$ pointwise in $[0,1]$, and is uniformly bounded ($|f_n(x)| \leq M$ for all x, all n), then $f_n \to f$ in L^1 norm. (And hence also in L^2 norm- as noted above, under these hypotheses, L^1 convergence and L^2 convergence are equivalent).

Proof. (*Reader beware*: The proof is recorded here for future reference, but following the details requires quite a bit more experience than I expect from the typical reader.) Try to understand the general idea, and feel free to ask me for more details if you are interested.)

It is enough to consider the case $f \equiv 0$. Pick $\epsilon > 0$. For each n, partition

$[0,1]$ into a 'good set' and a 'bad set', $[0,1] = G_n \sqcup B_n$:

$$B_n = \{x; |f_n| > \epsilon\}, \quad |f_n| \leq \epsilon \text{ in } G_n.$$

Denote by $|B_n|$ the measure ('generalized length') of the set B_n. If there is a subsequence n_i such that $|B_{n_i}|$ is bounded below (say, greater than some $\delta > 0$), then these bad sets cannot all be disjoint, and one may take a further subsequence n_i and find $x \in [0,1]$ so that $x \in B_{n_i}$ for all i, that is, $|f_{n_i}(x)| > \epsilon$ for all i, which is not possible if $f_n(x) \to 0$. Thus $|B_n| \to 0$, and since $|f_n| \leq M$ everywhere, we have:

$$\int_0^1 |f_n| = \int_{B_n} |f_n| + \int_{G_n} |f_n| \leq M|B_n| + \int_{G_n} |f_n| \leq M|B_n| + \epsilon \leq 2\epsilon,$$

for n sufficiently large. Hence $\limsup \int_0^1 |f_n| \leq 2\epsilon$, and since ϵ is arbitrary, it follows that $\int_0^1 |f_n| \to 0$.

3. Fourier convergence theorems.

Theorem 1. (Dirichlet 1824) Let $f \in C_{pw}$. Then $s_N \to \tilde{f}$ pointwise on $\mathbb{R}$, where

$$\tilde{f}(x) := (f(x_+) + f(x_-))/2$$

(average of the one-sided limits at x.)

Theorem 2. Let $f \in C_{per}$. Then $s_N \to f$ uniformly in $\mathbb{R}$.

Theorem 3. Let $f \in C_{pw}$. Then $s_N \to f$ in L^2 norm.

In practice Theorem 2 is the most powerful, but the proofs of all three results are interdependent in an interesting way. Be sure to review the definitions (above) of the spaces C_{per} and C_{pw}! Before we go on, do the following problem:

Exercise 2. Let f be 2π-periodic, equal to zero everywhere except at $2n\pi$, where it equals 1. (i) what are the Fourier coefficients of f? (ii) To what function do the partial sums $s_N(x)$ converge? Is this pointwise convergence? Uniform? In L^2 norm? Are the answers in accordance with theorems 1-3 above?

Decay of Fourier coefficients. The Fourier coefficients of a piecewise continuous function satisfy the obvious bound:

$$|a_n|, |b_n| \leq \frac{1}{\pi} \int_{\pi}^{\pi} |f(x)| dx.$$

If f is continuous, with piecewise-continuous first-derivative (i.e., $f \in C_{per}$) integration by parts and periodicity yield:

$$a_n[f] = \frac{1}{n} b_n[f'], \quad b_n[f] = -\frac{1}{n} a_n[f'],$$

and hence the bound:

$$|a_n|, |b_n| \leq \frac{1/\pi}{n} \int_{-\pi}^{\pi} |f'|(x) dx.$$

With f is 'more regular' (has more continuous derivatives), we may iterate this procedure and obtain:

$$f \in C_{per}^k \Rightarrow |a_n|, |b_n| \leq \frac{1/\pi}{n^{k+1}} \int_{-\pi}^{\pi} |f^{(k+1)}(x)| dx.$$

To go further, we have to introduce a basic calculation in the 'L^2 theory'. The symmetric bilinear form in C_{pw}:

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x) g(x) dx$$

has many of the properties of an 'inner product' (generalized 'dot product') in a finite-dimensional vector space. One crucial property fails: it is not 'positive-definite', but only 'positive semi-definite':

$$||f||^2 := \langle f, f, \rangle = 0 \Rightarrow f = 0 \text{ except possibly on a set of measure } 0.$$

(For an inner product, $||v||^2 = 0 \Rightarrow v = 0$). The usual problem that integration doesn't see sets of measure zero rears its ugly head. Still, the analogy is so useful that most people just ignore this, 'identify functions coinciding up to sets of measure zero' without stopping to think about the consequences (e.g. it makes no sense to talk about the value of a function at a point), call it an inner product anyway and forge on ahead! In this spirit, consider the functions:

$$e_0 = \frac{1}{\sqrt{2\pi}}, \quad e_n = \frac{1}{\sqrt{\pi}} \cos nx, \quad f_n = \frac{1}{\sqrt{\pi}} \sin nx.$$

It is very easy to see that the e_n $(n \geq 0)$ jointly with the f_n $(n \geq 1)$ define an 'orthonormal set' for the L^2 inner product. The Fourier coefficients have the expressions:

$$a_0 = \sqrt{2/\pi} \langle f, e_0 \rangle, \quad a_n = (1/\sqrt{\pi}) \langle f, e_n \rangle, \quad b_n = (1/\sqrt{\pi}) \langle f, f_n \rangle,$$

while the partial sum $s_N(x)$ can be written as:

$$s_N(x) = \sum_{n=0}^{N} \langle f, e_n \rangle e_n + \sum_{n=1}^{N} \langle f, f_n \rangle f_n,$$

which looks *exactly* like the formula for orthogonal projection onto a $2N+1$-dimensional subspace of a vector space with an inner product, in terms of an orthonormal basis for the subspace!

Encouraged by this, we compute the 'norm-squared' of the remainder using the usual expansion for a quadratic form and find (using orthogonality-*do try this at home!*):

$$||f - s_N||^2 = ||f||^2 - \sum_{n=0}^{N} \langle f, e_n \rangle^2 - \sum_{n=1}^{N} \langle f, f_n \rangle^2,$$

or in terms of Fourier coefficients:

$$||f||^2 - \pi[a_0^2 + \sum_{n=1}^{N} a_n^2 + b_n^2] = ||f - s_N||^2 \geq 0.$$

This innocent-looking calculation has several important consequences:

1. (*'Bessel's inequality'*): the series $\sum_{n\geq 0} a_n^2$ and $\sum_{n\geq 1} b_n^2$ are convergent:

$$a_0^2 + \sum_{n\geq 1} (a_n^2 + b_n^2) \leq \pi^{-1} ||f||^2.$$

2. (*'Parseval's equality'*):

$$s_N \to f \text{ in } L^2 \text{ norm } \Leftrightarrow \sum_{n\geq 0} a_n^2 + \sum_{n\geq 1} b_n^2 = \pi^{-1} ||f||^2.$$

3. (*'Riemann-Lebesgue lemma'*): $|a_n|$ and $|b_n|$ tend to zero as n tends to infinity, for any $f \in C_{pw}$.

This clearly follows from Bessel's inequality. We can apply this iteratively and get:

$$f \in C^k_{per} \Rightarrow n^{k+1} \max\{|a_n|, |b_n|\} \to 0,$$

a stronger decay condition than that seen above.

Even more, this is all we need to give the proofs of theorems 2 and 3, *assuming* theorem 1.

Proof of uniform convergence (Theorem 2). We assume theorem 1. Since, from theorem 1, we know $s_N(x) \to f(x)$ pointwise, we may write for each $x \in \mathbb{R}$:

$$|s_N(x) - f(x)| = |\sum_{n=N+1}^{\infty} a_n \cos nx + b_n \sin nx| \leq \sum_{N+1}^{\infty}(|a_n| + |b_n|).$$

To estimate the right-hand side, recall we are assuming f is piecewise C^1, so using the notation $a'_n = a_n[f'], b'_n = b_n[f']$, the right-hand side is bounded above by:

$$\sum_{n=N+1}^{\infty} \frac{1}{n}(|a'_n|+|b'_n|) \leq \sqrt{2}(\sum_{n=N+1}^{\infty} |a'_n|^2+|b'_n|^2)^{1/2} \leq \sqrt{2}(\sum_{n=N+1}^{\infty} \frac{1}{n^2})^{1/2}\frac{1}{\sqrt{\pi}}||f'||^{1/2},$$

where we used the Cauchy-Schwarz inequality for convergent series (and Bessel's inequality for f'). From the estimate for the 'tail' part of a convergent series given by the integral test:

$$\sum_{n=N+1}^{\infty} \frac{1}{n^2} \leq \int_N^{\infty} \frac{1}{x^2}dx = \frac{1}{N},$$

we get not only uniform convergence, but also a *rate of convergence*:

$$\sup_{x\in\mathbb{R}} |s_N(x) - f(x)| \leq (\frac{2}{\pi}||f'||)^{1/2}\frac{1}{\sqrt{N}}.$$

Proof of L^2 convergence (Theorem 3). We assume Theorem 2. The idea is to approximate f (which is only piecewise continuous) in L^2 norm by continuous functions, to which Theorem 2 can be applied. We need the following lemma:

Lemma. If f is piecewise continuous (say, in $[a, b]$), with piecewise continuous derivative, then given $\epsilon > 0$ we may find f_ϵ continuous, with piecewise continuous derivative, so that $||f_\epsilon - f|| < \epsilon$.

In lieu of a proof, you are asked in Exercise 3 (below) to show this in a simple case (which contains the main idea for the general case)

Given $\epsilon > 0$, use the lemma to find f_ϵ in C_{per} with

$$||f - f_\epsilon|| \leq \epsilon/4$$

(L^2 norm). Then use Theorem 2 (and the fact that uniform convergence implies convergence in L^2) to find N so that f_ϵ is well-approximated in L^2 norm by the Nth. partial sum of its Fourier series:

$$||f_\epsilon - s_N[f_\epsilon]|| \leq \epsilon/4.$$

Finally, we need an estimate for the L^2 norm of $s_N[f_\epsilon] - s_N[f]$. Note that this is none other than the Nth. partial sum of the Fourier series of $f_\epsilon - f$, to which the identity above (for the L^2 norm of the remainder) can be applied, yielding:

$$||f_\epsilon - f - s_N[f_\epsilon - f]|| \leq ||f_\epsilon - f|| \leq \epsilon/4.$$

Using the triangle inequality, we have:

$$||s_N[f_\epsilon - f]|| \leq 2||f_\epsilon - f|| \leq \epsilon/2.$$

Putting everything together, we find:

$$||f - s_N[f]|| \leq ||f - f_\epsilon|| + ||f_\epsilon - s_N[f_\epsilon]|| + ||s_N[f_\epsilon] - s_N[f]|| \leq \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{2} = \epsilon,$$

proving L^2 convergence (since ϵ is arbitrary).

Exercise 3. Let $f(x)$ be the piecewise continuous function in $[-\pi, \pi]$:

$$f(x) = 1 \text{ for } x \in (0, \pi], \quad f(x) = -1 \text{ for } x \in [-\pi, 0).$$

Given $h > 0$, define f_h in $[-\pi, \pi]$ by:

$$f_h(x) = \frac{x}{h} \text{ for } |x| \leq h, \quad f_h(x) = f(x) \text{ otherwise.}$$

(i) Sketch a graph of f and f_h, verifying that f_h is continuous; (ii) show (directly) that $||f - f_h|| \to 0$ as $h \to 0$.

4. Proof of Dirichlet's theorem.

The first step in the proof is a computation. Given $f \in C_{pw}$, the Nth. partial sum of its Fourier series can be written as:

$$s_N(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} (\frac{1}{2} + \sum_{n=1}^{N} \cos nx \cos ny + \sin nx \sin ny) f(y) dy.$$

Using a standard trigonometric formula, we find an expression for $s_N(x)$ in 'convolution form':

$$s_N(x) = \int_{-\pi}^{\pi} K_N(x - y) f(y) dy,$$

with $K_N(x)$ (the *Dirichlet kernel*) defined by:

$$K_N(x) = \frac{1}{\pi}(\frac{1}{2} + \sum_{n=1}^{N} \cos(ny)).$$

Amazingly, this sum can be computed explicitly! Using the De Moivre-Laplace formula to represent $\cos(nx)$ as a sum of complex exponentials, we find that $K_N(x)$ is (for $x \neq 0$) the sum of $2N+1$ terms of a geometric sequence with ratio e^{ix}:

$$K_N(x) = \frac{1}{2\pi} \sum_{n=-N}^{N} e^{nix} = \frac{e^{(N+1)ix} + e^{-Nix}}{e^{ix} - 1}, \quad x \neq 0,$$

and the last expression can be written as a ratio of sines (multiplying numerator and denominator by $e^{-ix/2}$, yielding:

$$K_N(x) = \frac{1}{\pi} \frac{\sin[(N+1/2)x]}{2\sin(x/2)}, \text{ for } x \neq 0; \quad K_N(0) = \frac{1}{\pi}(2N+1).$$

Note that it follows directly from the definition that:

$$\int_{-\pi}^{\pi} K_N(x)dx = 1,$$

and by a change of variables:

$$s_N(x) = \int_{-\pi}^{\pi} f(x+y)K_N(y)dy.$$

Fixing $x \in [-\pi, \pi]$, assume that f is continuous and differentiable at x (the proof is given only in this case). Then:

$$s_N(x) - f(x) = \int_{-\pi}^{\pi} [f(x+y) - f(x)]K_N(y)dy = \frac{1}{\pi} \int_{-\pi}^{\pi} g_x(y) \sin[(N + \frac{1}{2})y]dy,$$

where:

$$g_x(y) = \frac{f(x+y) - f(x)}{2\sin y/2}, \quad y \neq 0; \quad g_x(0) = f'(x).$$

Note that g_x is continuous and 2π-periodic. Now using:

$$\sin(N+1/2)y = \sin(Ny)\cos(y/2) + \cos(Ny)\sin(Ny/2),$$

we see that the integral may itself be identified as a sum of Fourier coefficients! (Namely, for $c_x : y \mapsto g_x(y)\cos(y/2)$ and $s_x : y \mapsto g_x(y)\sin(y/2)$:)

$$s_N(x) = b_N[c_x] + a_N[s_x],$$

and therefore tends to zero, by the Riemann-Lebesgue lemma.

Exercise 4. [Strauss, p.129] Consider the sequence of functions $g_n(x)$ in $[0,1]$ defined by: for n *even*:

$$g_n(x) = 1 \text{ in } [3/4 - 1/n^2, 3/4 + 1/n^2), \quad g_n = 0 \text{ elsewhere;}$$

for n *odd*:

$$g_n(x) = 1 \text{ in } [1/4 - 1/n^2, 1/4 + 1/n^2) \quad g_n = 0 \text{ elsewhere.}$$

Show that $g_n(x) \to 0$ in L^2 norm, but not in the pointwise sense.

Exercise 5. [Strauss, p. 130]. Consider the Fourier *sine* series of each of the functions below in $[0,\pi]$. Use the theorems above to discuss convergence of each of them in the uniform, pointwise and L^2 senses (in $[0,\pi]$). (a) $f(x) = x^3$; (b) $f(x) = \pi x - x^2$; (c) $f(x) = x^2 + 1$. Don't waste your time actually computing the Fourier coefficients; draw some informative diagrams instead.

Exercise 6. [Strauss, p. 140] (*Wirtinger's inequality)* Show that if f is a C^1 function in $[-\pi,\pi]$ and $\int_{-\pi}^{\pi} f(x)dx = 0$, then:

$$\int_{-\pi}^{\pi} |f|^2 dx \le \int_{-\pi}^{\pi} |f'|^2(x)dx.$$

Hint: Use Parseval's equality!

Later (in class) we will see a remarkable application of this fact.