SPHERICAL HARMONICS AND HOMOGENEOUS HARMONIC POLYNOMIALS

1. The spherical Laplacean. Denote by $S \subset \mathbb{R}^3$ the unit sphere. For a function $f(\omega)$ defined on S, let $\tilde{f}$ denote its extension to an open neighborhood $\mathcal{N}$ of S, constant along normals to S (i.e., constant along rays from the origin). We say $f \in C^2(S)$ if $\tilde{f}$ is a C^2 function in $\mathcal{N}$, and for such functions define a differential operator Δ_S by:

$$\Delta_S f := \Delta \tilde{f},$$

where Δ on the right-hand side is the usual Laplace operator in $\mathbb{R}^3$. With a little work (omitted here) one may derive the expression for Δ in polar coordinates (r, ω) in $\mathbb{R}^2$ $(r > 0, \omega \in S)$:

$$\Delta u = u_{rr} + \frac{2}{r} u_r + \frac{1}{r^2} \Delta_S u.$$

(Here $\Delta_S u(r, \omega)$ is the operator Δ_S acting on the function $u(r, .)$ in S, for each fixed r.)

A *homogeneous polynomial* of degree $n \geq 0$ in three variables (x, y, z) is a linear combination of 'monomials of degree n':

$$x^{d_1} y^{d_2} z^{d_3}, \quad d_i \geq 0, \quad d_1 + d_2 + d_3 = n.$$

This defines a vector space (over $\mathbb{R}$) denoted $\mathcal{P}_n$. A simple combinatorial argument (involving balls and separators, as most of them do), seen in class, yields the dimension:

$$d_n := dim(\mathcal{P}_n) = \frac{1}{2}(n+1)(n+2).$$

Writing a polynomial $p \in \mathcal{P}_n$ in polar coordinates, we necessarily have:

$$p(r, \omega) = r^n f(\omega), \quad f = p_{|S},$$

where f is the restriction of p to S. This is an injective linear map $p \mapsto f$, but the functions on S so obtained are rather special (a d_n-dimensional subspace of the infinite-dimensional space $C(S)$ of continuous functions-let's call it $\mathcal{P}_n(S)$)

We are interested in the subspace $\mathcal{H}_n \subset \mathcal{P}_n$ of homogeneous harmonic polynomials of degree n $(\Delta p = 0)$.

Lemma. Let $p \in \mathcal{P}_n$; write $p = r^n f(\omega)$, as above. Then:

$$p \in \mathcal{H}_n \Leftrightarrow \Delta_S f + n(n+1)f = 0.$$

Proof. Using the above expression for Δ, one finds easily:

$$\Delta(r^n f(\omega)) = n(n+1)r^{n-2}f(\omega) + r^{n-2}\Delta_S f(\omega),$$

for $n \geq 2$ and, for $n = 1$:

$$\Delta(rf(\omega)) = \frac{2}{r}f(\omega) + \frac{1}{r}\Delta_S f(\omega).$$

The restrictions to S of elements of $\mathcal{H}_n$ are known as 'spherical harmonics of degree n', and are therefore eigenfunctions of Δ_S with eigenvalue $n(n+1)$, a subspace of $C^2(S)$ denoted $\mathcal{H}_n(S)$. Thus, restriction to S defines an injective linear map:

$$p \mapsto Y = f_{|S}, \quad \mathcal{H}_n \to \mathcal{H}_n(S).$$

(*Remark:* This map is actually an isomorphism, but this is not obvious. If you start with a spherical harmonic $Y(\omega)$ and set $p = r^n Y(\omega)$ (in polar coordinates), it is not clear that p is a polynomial- all we know is that it is a harmonic function in $\mathbb{R}^3$ (minus the origin) with 'growth rate' r^n (that is, $|p(r, \omega)| \leq Cr^n$ for some constant $C > 0$.) And it turns out one can show that any harmonic function in $\mathbb{R}^3$ with this growth rate must be a polynomial (of degree at most n), but this takes some work!)

2. Orthogonal decompositions. It is easy to write down a basis for $\mathcal{P}_n$. For instance, for $n = 3$:

$$\mathcal{P}_n = span\{x^3, y^3, z^3, x^2y, x^2z, xy^2, y^2z, xz^2, yz^2, xyz\}.$$

How do we find a basis for $\mathcal{H}_n$? How do we even compute its dimension? It turns out the answer involves a beautiful application of basic linear algebra. To start, define an inner product on $\mathcal{P}_n$ by:

$$\langle x^{d_1}y^{d_2}z^{d_3}, x^{m_1}y^{m_2}z^{m_3}\rangle_n := d_1!d_2!d_3! \quad \text{if } d_1 = m_1, d_2 = m_2, d_3 = m_3,$$

and equal to zero otherwise. This just says that the different monomials that define a natural basis for $\mathcal{P}_n$ are taken to be pairwise orthogonal, with length squared depending on the degrees of each variable. (The occurrence of factorials is explained in Section 3.)

Consider the linear map:

$$T : \mathcal{P}_n \to \mathcal{P}_{n+2}, \quad T(p) = (x^2 + y^2 + z^2)p.$$

We also have a linear map going the other way, the Laplacean:

$$\Delta : \mathcal{P}_{n+2} \to \mathcal{P}_n, \quad \Delta q = q_{xx} + q_{yy} + q_{zz}.$$

Note that T is injective (i.e., its nullspace is $\{0\}$.) It turns out there is a surprising connection between these maps.

Theorem. $\mathcal{P}_{n+2} = T(\mathcal{P}_n) \oplus \mathcal{H}_{n+2}$. (orthogonal direct sum, with respect to the inner product defined above.)

Since $T(\mathcal{P}_n)$ is isomorphic to $\mathcal{P}_n$, this gives the dimension of $\mathcal{H}_n$:

Corollary. $dim\mathcal{H}_n = d_n - d_{n-2} = 2n + 1$ (for $n \geq 2$); $dim\mathcal{H}_1 = 3$ is obvious.)

In particular, assuming the fact about harmonic functions in $\mathbb{R}^3$ described in the Remark, this is also the dimension of the space $\mathcal{H}_n(S)$ of spherical harmonics, or the multiplicity of $n(n+1)$ as an eigenvalue of Δ_S.

Recall the definition of 'adjoint map' from linear algebra: if E, F are vector spaces endowed with inner products $\langle ., . \rangle_E, \langle ., . \rangle_F$ and $T : E \to F$ is a linear map, the adjoint $T^* : F \to E$ is defined by the requirement:

$$\langle T^* v, w \rangle_E = \langle v, Tw \rangle_F,$$

for all vectors $v \in F, w \in E$. The theorem follows from the following fact:

Lemma: Let $T : \mathcal{P}_n \to \mathcal{P}_{n+2}$ be the linear map defined above. With respect to the inner products defined earlier on $\mathcal{P}_n, \mathcal{P}_{n+2}$, we have: $T^* = \Delta$.

Proof. It is enough to show that, for the basic monomials in x, y, z of degree n and $n + 2$:

$$\langle T(x^{d_1} y^{d_2} z^{d^3}), x^{m_1} y^{m_2} z^{m_3} \rangle_{n+2} = \langle x^{d_1} y^{d_2} z^{d^3}, \Delta(x^{m_1} y^{m_2} z^{m_3}) \rangle_n.$$

This is an easy calculation, left to the reader.

Given the lemma, the theorem follows directly from the well-known result in linear algebra: if $T : E \to F$ is linear,

$$F = T(E) \oplus Ker(T^*),$$

orthogonal direct sum with respect to $\langle .,. \rangle_F$. 'Ker' denotes nullspace (or 'kernel'), in particular $Ker(\Delta) = \mathcal{H}_{n+2}$. Applying the theorem repeatedly, we have the isomorphisms:

Corollary 1.

$$\mathcal{P}_n = \mathcal{H}_n \oplus r^2\mathcal{H}_{n-2} \oplus r^4\mathcal{H}_{n-4} \oplus \dots .$$

(Here we use $r^2\mathcal{H}_n$ to denote $T(\mathcal{H}_n)$.) Restricting to the unit sphere, this gives:

Corollary 2.

$$\mathcal{P}_n(S) = \mathcal{H}_n(S) \oplus \mathcal{H}_{n-2}(S) \oplus \mathcal{H}_{n-4}(S) \oplus \dots .$$

(Of course, both direct sums are finite.)

Corollary 1 means any polynomial of degree n in three variables can be expressed uniquely as a linear combination of terms of the form: r^{2k} times a harmonic polynomial of degree $n-2k$ (this generalizes to any number of variables.)

There is a more important theoretical consequence, but it depends on the 'Weierstrass density theorem', which is in the same general circle of ideas. It says that any continuous function in $\mathbb{R}^3$ can be uniformly approximated (over any given compact set, say the unit ball) by polynomials. This is a fundamental theorem, so we include a proof in the appendix. Using corollary 2 and the Weierstrass density theorem we conclude:

Corollary 3. *Any continuous function on the unit sphere S can be approximated uniformly on S (and therefore in $L^2(S)$) by spherical harmonics.*

In corollary 3, by spherical harmonics we understand restrictions to S of (not necessarily homogeneous) harmonic polynomials in $\mathbb{R}^3$, of increasing degree.

One needs to be careful here: given any continuous function $f \in C(S)$, one can form its 'generalized Fourier series' in terms of eigenfunctions of Δ_S (=spherical harmonics), with coefficients defined by integration in the usual way. Corollary 3 *does not say* that the partial sums of this 'Fourier series' converge uniformly to S (this would be false for general continuous functions, just as for periodic functions of 1 variable.) The fact that Corollary 3 is silent on how an approximation may be found limits its usefulness (for solving PDE via eigenfunction expansions.)

On a more practical level, the theorem gives an *algorithm* to obtain a basis for $\mathcal{H}_n$ from a basis of $\mathcal{P}_n$. Consider, for example, the basis given above for $\mathcal{P}_3$. We know that $dim\mathcal{H}_3 = 7$, and four linearly independent elements can be written down 'by inspection':

$$\{xy^2 - xz^2, x^2y - yz^2, x^2z - y^2z, xyz\}.$$

To find three others, the theorem says that if we start from an element of $\mathcal{P}_3$ and subtract from it its orthogonal projection onto $T(\mathcal{P}_1)$, we get an element of $\mathcal{H}_3$. It is easy to write down an *orthogonal* basis for $T(\mathcal{H}_1)$:

$$T(\mathcal{H}_1) = span\{p_1 = x^3+xy^2+xz^2, p_2 = x^2y+y^3+yz^2, p_3 = x^2z+y^2z+z^3\}.$$

It is easy to see that $|p_1|^2 = |p_2|^2 = |p_3|^2 = 10$. So the orthogonal projection of $p \in \mathcal{P}_3$ onto $\mathcal{H}_3$ is given by:

$$proj(p) = p - \frac{1}{10}(\langle p, p_1\rangle p_1 + \langle p, p_2\rangle p_2 + \langle p, p_3\rangle p_3).$$

Consider $p = x^3$. Clearly $\langle p, p_2\rangle = \langle p, p_3\rangle = 0$, while $\langle p, p_1\rangle = 6$. Thus:

$$proj(x^3) = x^3 - \frac{6}{10}(x^3 + xy^2 + xz^2) = \frac{1}{5}(2x^3 - 3xy^2 - 3xz^2),$$

so we find that $2x^3 - 3xy^2 - 3xz^2 \in \mathcal{H}_3$ (as the reader may wish to check directly). Permuting the variables, we complete a basis of $\mathcal{H}_3$ with $-3x^2y + 2y^3 - 3yz^2$ and $-3x^2z - 3y^2z + 2z^3$.

3. A coordinate-free approach and Legendre polynomials.

Remark. This section is written at a more advanced level than that expected for the average student in this course; it will be expanded later.

The space $\mathcal{P}_n$ is invariant under linear changes of the coordinates (x, y, z), so we expect it has a coordinate-free description. Indeed, if E is real vector space we can define $\mathcal{P}_n(E)$ as the space of 'totally symmetric multilinear forms of degree n', and this is isomorphic to the space of homogeneous degree n polynomials, for any fixed linear coordinate system in E.

This description may be used to explain the 'factorial weights' in the definition of inner product on $\mathcal{P}_n$. An inner product on E induces in a natural way one in the space $L_n(E)$ of multilinear forms of degree n: if (e_i) is an orthonormal basis of E^*, an o.n. basis of L_n is given by the forms $e_{i_1} \otimes \ldots \otimes e_{i_n}$. We want to use this to define an inner product on $\mathcal{P}_n(E)$, but there is the problem that the natural projection π from L_n to $\mathcal{P}_n$ is not an isomorphism. The solution is to pick an orthogonal complement $S \subset L_n$ to its kernel and define the inner product on $\mathcal{P}_n$ so that $\pi_{|S}$ is an isometry, up to a constant C depending only on n. Given an orthonormal basis, there is a natural way to pick S. For example, if $dim E = 3$, natural orthonormal basis vectors for S and their images under π are:

$$\frac{1}{\sqrt{3}}(e_1 \otimes e_1 \otimes e_2 + e_1 \otimes e_2 \otimes e_1 + e_2 \otimes e_1 \otimes e_1) \mapsto \sqrt{3} e_1^2 e_2;$$

$$e_1 \otimes e_1 \otimes e_1 \mapsto e_1^3;$$

$$\frac{1}{\sqrt{6}}(e_1 \otimes e_2 \otimes e_3 + e_3 \otimes e_1 \otimes e_2 + \ldots) \mapsto \sqrt{6} e_1 e_2 e_3.$$

(There are 6 basis vectors of the first type, 3 of the second and one of the third. In the third vector, the sum is over all six permutations of e_1, e_2, e_3.)

Since the vectors on the left (in L_n) have length one, we see that the squared lengths of the vectors on the right (in $\mathcal{P}_n$) must satisfy:

$$3|e_1^2 e_2|^2 = |e_1^3|^2 = 6|e_1 e_2 e_3|^2 = C.$$

A natural choice (making all squared lengths integers) is $C = 6$, and then:

$$|e_1^2 e_2|^2 = 2, \quad |e_1^3|^2 = 6, \quad |e_1 e_2 e_3|^2 = 1.$$

In general, the number of vectors in $L_n(E)$ projecting to a given vector $e_1^{d_1} e_2^{d_2} e_3^{d_3}$ in $\mathcal{P}_n(E)$ is: $n!/(d_1!d_2!d_3!)$ (combinatorics again!). This explains the factorial weights.

Legendre polynomials. Given a particular choice of coordinates in $\mathbb{R}^3$, it is useful for certain applications to be able to write down explicitly a basis for $\mathcal{H}_n(S)$. This is more transparent if we introduce partly complex coordinates (w, t) in $\mathbb{R}^3$, where $t \in \mathbb{R}$ and $w \in \mathbb{C}$, and we think of the $\mathbb{C}$ plane as perpendicular to the t axis. (In terms of the coordinates used earlier, $w = x + iy$ and $t = z$.) On the unit sphere, $t^2 + |w|^2 = 1$. It turns out any $Y \in \mathcal{H}_n(S)$ can be written as a linear combination of the form:

$$Y(t, w) = c_0 P_n(t) + \sum_{m=1}^{n} (c_m w^m + \bar{c}_m \bar{w}^m) P_n^m(t), \quad t^2 + |w|^2 = 1,$$

where $c_0 \in \mathbb{R}$, $c_m \in \mathbb{C}$. Here $P_n(t)$ and $P_n^m(t)$ are polynomials in t, of degree n and $n - m$, respectively, known as ‘Legendre polynomials’. Ordinarily they are defined via differential equations. Our next goal is to explain how the above 'orthogonal projection method' can be used to compute these polynomials explicitly, using only linear algebra.

Another way to put this is that there is a basis of $\mathcal{H}_n(S)$ of the form:

$$\{Re[w^n], Im[w^n], Re[w^{n-1}]P_n^{n-1}(t), Im[w^{n-1}]P_n^{n-1}(t), \dots, Re[w]P_n^1(t), Im[w]P_n^1(t), P_n(t)\}.$$

Let's check this first when $n = 2$: certainly the functions:

$$w^2, \quad \bar{w}^2, \quad wt, \quad \bar{w}t$$

are in $\mathcal{H}_2(S)$ (more precisely: their real and imaginary parts are). To form a basis, we need a fifth element (we know the dimension is 5). In $\mathbb{R}^3$ the polynomial $p = 2z^2 - (x^2 + y^2)$ would do the job; on the unit sphere, this is is the same as $3z^2 - 1$ (since $x^2 + y^2 = 1 - z^2$); so $P_2(t) = 3t^2 - 1$ is our fifth basis element, and we may also let $P_2^1(t) = t, P_2^2(t) = 1$.

For $n = 3$, we need 7 basis elements, and some of them are easy to list:

$$\{w^3, \quad \bar{w}^3, \quad w^2t, \quad \bar{w}^2t, \quad wP_3^1(t), \quad \bar{w}P_3^1(t), \quad P_3(t)\}.$$

From the earlier discussion, we know that $p = 2z^3 - 3(x^2 + y^2)z \in \mathcal{H}_3$ in $\mathbb{R}^3$. But on S, this is the same as $5z^2 - 3z$, so we set:

$$P_3(t) = 5t^3 - 3t.$$

Searching for a basis element linear in x, consider $p = x(2x^2 - 3y^2 - 3z^2)$, found earlier. The problem is that x^2 and y^2 have different coefficients; to remedy that, we add to p the harmonic polynomial $x(5y^2 - 5z^2)$:

$$2x^3 - 3xy^2 - 3xz^2 + x(5y^2 - 5z^2) = 2x(x^2 + y^2) - 8xz^2,$$

and on the unit sphere this equals $2x(1 - z^2) - 8xz^2$, or $2x(1 - 5z^2)$. Thus the remaining elements of the basis of $\mathcal{H}_3(S)$ are (the real and imaginary parts of):

$$(5t^2 - 1)w, \quad (5t^2 - 1)\bar{w},$$

and we may set:

$$P_3^1(t) = 5t^2 - 1.$$

Remark: In general, $P_n^m(t)$ is the $m - th$ order derivative of $P_n(t)$ (up to a constant).

Exercise. In this exercise, we outline the steps needed to find $P_4(t)$ using the projection method.

The goal is to find the projection of z^4 onto $\mathcal{H}_4$. Given the orthogonal decomposition:

$$\mathcal{P}_4 = r^2\mathcal{H}_2 \oplus r^4\mathcal{H}_0,$$

we see that we only need to consider the polynomials in $\mathcal{P}_4$:

$$p_1 = (x^2+y^2+z^2)(z^2-x^2), \quad p_2 = (x^2+y^2+z^2)(z^2-y^2), \quad q = (x^2+y^2+z^2)^2.$$

(i) Show that $|p_1|^2 = |p_2|^2 = 56$, $|q|^2 = 120$ and $\langle p_1, p_2 \rangle = 28$.

(ii) Apply the Gram-Schmidt orthogonalization method to find the polynomial $\tilde{p}_2 \in r^2\mathcal{H}_2$ orthogonal to p_1:

$$\tilde{p}_2 = \frac{1}{2}(x^2 + y^2 + z^2)(z^2 - 2y^2 + x^2),$$

with $|\tilde{p}_2|^2 = 42$.

(iii) Show that the orthogonal projection of z^4 onto $\mathcal{H}_4$ is the polynomial:

$$h = z^4 - \frac{3}{7}p_1 - \frac{2}{7}\tilde{p}_2 - \frac{1}{5}q.$$

(iv) Verify directly that h is indeed harmonic (first show that $\Delta p_1 = 14(z^2 - x^2), \Delta \tilde{p}_2 = 7(x^2 - 2y^2 + z^2)$ and $\Delta q = 20(x^2 + y^2 + z^2)$).

(v) Show that on the unit sphere we have:

$$h_{|S} = z^4 - \frac{2}{7}(3z^2 - 1) - \frac{1}{5}.$$

Thus we may take for $P_4(t)$:

$$P_4(t) = 35t^4 - 30t^2 + 3.$$

Remark: Up to a constant, $P_n(t)$ is the n-th derivative of $(t^2 - 1)^n$.