MATH 435- Homework-due 4/16/2002

1. Find a simple formula for the solution of the three-dimensional heat equation $u_t - \Delta u = 0$, with initial condition:

$$f(x, y, z) = xy^2 z.$$

Hint: Look for a solution of the form $u(x, y, z, t) = xv(t)z$, where $v(t)$ solves a first-order ODE, $v(0) = y$.

2. Find the solution of the heat equation in a half space $\{(x, y, z); z > 0\}$ with Neumann boundary condition $\partial u/\partial z = 0$ when $z = 0$, and given initial condition $u(x, y, z, 0) = f(x, y, z)$. The solution should be an integral involving the heat kernel. *Hint:* Use the method of reflection.

3. Let $D \subset \mathbb{R}^3$ be the ball of radius 1 centered at 0. Consider the nonhomogeneous heat equation $u_t - \Delta u = 1$ in D, with $u = 0$ on the boundary ∂D (all t) and $u \equiv 0$ in D at $t = 0$.

(i)Show the problem has a steady-state solution (i.e., independent of t)

$$u(r) = (1 - r^2)/6.$$

(ii)Use the maximum principle to show: $u(x, t) \leq (1 - ||x||^2)/6$ for all t.

(iii)Find the largest value of a and the largest value of b for which the minimum principle guarantees the inequality:

$$b(1 - e^{-at})\frac{1 - ||x||^2}{6} \leq u(x, t),$$

for all $x \in D, t \geq 0$.

4.Show there is at most one solution to the Neummann problem for the heat equation:

$$u_t - \Delta u = h(x, t) \text{ in } D, \quad \partial u/\partial n = 0 \text{ on } \partial D, \quad u(x, 0) = 0,$$

for a bounded domain $D \subset \mathbb{R}^3$.

*Hint:*Let w be the difference of two solutions. Show that $I(t) = \int_D w^2(x, t)dx$ is non-increasing in t.

5.Find the dimension of each of the following vector spaces:

(i) The space of all solutions of $u'' + x^2 u = 0, u = u(x)$;

(ii)The eigenspace with eigenvalue $(2\pi/L)^2$ of the operator d^2/dt^2 on the interval $[-L, L]$, with periodic boundary consitions;

(iii)The eigenspace of harmonic functions in the unit disk in $\mathbb{R}^2$, with homogeneous Neumann boundary conditions;

(iv)The eigenspace with eigenvalue $\lambda = 25\pi^2$ of Δ in the unit square $[0, 1]^2 \subset \mathbb{R}^2$, with homogeneous Dirichlet boundary consitions;

(v)The space of all solutions of the wave equation $u_{tt} = u_{xx}$ on the real line.