

## NON-ELLIPTIC STRUCTURE OF THE RICCI OPERATOR.

**0. Quasilinear structure and linearization.** In local coordinates, we have the expressions for Ricci curvature and the Christoffel symbols of the Levi-Civita connection:

$$R_{jk} = \Gamma_{jk|m}^m - \Gamma_{mk|j}^m - \Gamma_{jk}^n \Gamma_{nm}^m - \Gamma_{mk}^n \Gamma_{nj}^m,$$

$$\Gamma_{jk}^m = \frac{1}{2} g^{ml} (g_{jl|k} + g_{lk|j} - g_{jk|l})$$

(bars denote partial derivatives in the given coordinate system.) To obtain useful global expressions, R.Schoen observed one can consider a ‘background metric’  $\bar{g}$  on  $M$  with its Levi-Civita connection  $\bar{\nabla}$  and the difference of connections, a symmetric  $(2, 1)$  tensor on  $M$ :

$$D(X, Y) = \nabla_X^g Y - \bar{\nabla}_X Y.$$

It then turns out that the expressions of the tensorial components of  $D$  in terms of  $g$ , and of the Ricci tensor in terms of  $D$ , are familiar ones:

$$D_{jk}^m = \frac{1}{2} g^{ml} (\bar{\nabla}_k g_{jl} + \bar{\nabla}_j g_{lk} - \bar{\nabla}_l g_{jk}),$$

$$R_{jk} = \bar{R}_{jk} + \bar{\nabla}_m D_{jk}^m - \bar{\nabla}_j D_{mk}^m - D_{jk}^n D_{nm}^m - D_{mk}^n D_{nj}^m.$$

(To see why this is true, just compute in local normal coordinates for  $\bar{g}$ !)

From now on all components are tensorial (as opposed to local coordinate components.) The terms involving second covariant derivatives of the metric are:

$$\frac{1}{2} g^{ml} (-\bar{\nabla}_m \bar{\nabla}_l g_{jk} + \bar{\nabla}_m \bar{\nabla}_k g_{jl} + \bar{\nabla}_m \bar{\nabla}_j g_{lk} + \bar{\nabla}_j \bar{\nabla}_l g_{mk} - \bar{\nabla}_j \bar{\nabla}_k g_{ml} - \bar{\nabla}_j \bar{\nabla}_m g_{lk}).$$

The first term is a kind of Laplacian. The third and sixth terms add up to a commutator of covariant derivatives, a lower-order (curvature) term- so we drop them. To make the expression formally symmetric in  $j, k$ , we change the order of  $\bar{\nabla}_l$  and  $\bar{\nabla}_j$  in the fourth term (again at the cost of a lower-order term). With these changes, to identify the remaining terms (starting at the second) we write (commuting covariant derivatives when needed):

$$\begin{aligned} g^{ml} (\bar{\nabla}_m \bar{\nabla}_k g_{jl} + \bar{\nabla}_l \bar{\nabla}_j g_{mk} - \bar{\nabla}_j \bar{\nabla}_k g_{ml}) &= g^{ml} [\bar{\nabla}_k (D_{ml}^p g_{jp}) + \bar{\nabla}_j (D_{ml}^p g_{kp})] \\ &= g_{jp} \bar{\nabla}_k (g^{ml} D_{ml}^p) + g_{kp} \bar{\nabla}_j (g^{ml} D_{ml}^p) + \text{lower order} \end{aligned}$$

$$\begin{aligned}
&= g_{jp} \nabla^g_k (g^{ml} D_{ml}^p) + g_{kp} \nabla^g_j (g^{ml} D_{ml}^p) + \text{lower order} \\
&\text{(using } \bar{\nabla}_j X^p = \nabla_j^g X^p - D_{jq}^p X^q) \\
&= (L_W g)_{jk} + Q(g, \bar{\nabla} g)_{jk},
\end{aligned}$$

where  $L_W g$  is the Lie derivative of the metric (see below) with respect to the vector field:

$$W^p = g^{ml} D_{ml}^p$$

and  $Q(g, \bar{\nabla} g)$  is a symmetric  $(2, 0)$ -tensor, quadratic in covariant derivatives of  $g$  (with respect to  $\bar{\nabla}$ .) We conclude:

**Proposition.** (De Turck/Schoen). Given a ‘background metric’  $\bar{g}$  the Ricci tensor of  $g$  (as a differential operator on  $g$ ) admits the global expression:

$$Rc^g = -\frac{1}{2} tr_g \bar{\nabla}^2 g + \frac{1}{2} L_W g + Q(g, \bar{\nabla} g) + Rc^{\bar{g}},$$

where  $W$  is the vector field  $W = tr_g D$ , with  $D$  the symmetric  $(2, 1)$  tensor  $\nabla^g - \bar{\nabla}$ .

This exhibits  $Rc^g$  as a quasilinear tensorial second-order differential operator of  $g$ , for a fixed background connection.

The ‘background metric’ approach yields a fairly painless way to compute the linearization of the Ricci tensor. To find the linearization of  $-2Rc$  at a metric  $\bar{g}$  in the direction of  $h \in S_M^2$ , we consider the path of metrics  $g_s = \bar{g} + sh$  ( $|s| < \epsilon$ ) and compute the derivative:

$$-2D(Rc(\bar{g}))[h] = -2 \frac{d}{ds} \Big|_{s=0} Rc(g_s).$$

For this we use the expression found above:

$$-2R_{jk}^{g_s} = -2(\bar{\nabla}_m D_{jk}^m(s) - \bar{\nabla}_j D_{mk}^m(s)) - 2R_{jk}^{\bar{g}} + (D_s * D_s)_{jk}.$$

Differentiating in  $s$  we find, at  $s = 0$  (using  $D_0 \equiv 0$ ):

$$-2 \frac{d}{ds} \Big|_{s=0} R_{jk}^{g_s} =$$

$$g^{ml} (\bar{\nabla}_m \bar{\nabla}_l h_{jk} - \bar{\nabla}_m \bar{\nabla}_k h_{jl} - \bar{\nabla}_m \bar{\nabla}_j h_{lk} - \bar{\nabla}_j \bar{\nabla}_l h_{mk} + \bar{\nabla}_j \bar{\nabla}_k h_{ml} + \bar{\nabla}_j \bar{\nabla}_m h_{lk}).$$

(There are no other terms, since e.g.  $\bar{\nabla}_l \bar{\nabla}_m \bar{g}_{jk} = 0$ .) In this expression we perform the same commutations of covariant derivatives described earlier, this time keeping track of the curvature terms involved:

$$-2D(R^{\bar{g}})[h]_{jk} = g^{ml} (\bar{\nabla}_m \bar{\nabla}_l h_{jk} - \bar{\nabla}_m \bar{\nabla}_k h_{jl} - \bar{\nabla}_l \bar{\nabla}_j h_{mk} + \bar{\nabla}_j \bar{\nabla}_k h_{ml})$$

$$-g^{ml}(\bar{R}_{jlm}{}^q h_{qk} + \bar{R}_{jlk}{}^q h_{mq} + \bar{R}_{mjl}{}^q h_{qk} + \bar{R}_{mj k}{}^q h_{lq}),$$

and one can easily check that the curvature terms cancel!

Changing  $\bar{g}$  to an arbitrary metric  $g$ , we have the formula:

$$-2D(Rc(g))[h] = g^{ml}(\nabla_m \nabla_l h_{jk} - \nabla_m \nabla_k h_{jl} - \nabla_l \nabla_j h_{mk} + \nabla_j \nabla_k h_{ml})$$

### 1. The Bianchi identities.

*Bianchi I:*

$$R(X, Y)Z + R(Z, X)Y + R(Y, Z)X = 0$$

Using the fact the connection is torsion-free, the left-hand side equals:

$$[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]],$$

and this vanishes (Jacobi identity).

*Remark.* Bianchi I is used to show that the curvature operator on 2-forms  $\mathbf{Rm} : \Omega_M^2 \rightarrow \Omega_M^2$  is symmetric:

$$\langle \mathbf{Rm}(X \wedge Y), Z \wedge W \rangle = \langle R(X, Y)Z, W \rangle = \langle X \wedge Y, \mathbf{Rm}(Z \wedge W) \rangle.$$

*Bianchi II:*

$$\nabla_X R(Y, Z)W + \nabla_Y R(Z, X)W + \nabla_Z R(X, Y)W = 0.$$

Computing the first term on the left (at a given point  $p \in M$ ):

$$\begin{aligned} (\nabla_X R)(Y, Z)W &= \nabla_X([\nabla_Y, \nabla_Z]W) - [\nabla_Y, \nabla_Z](\nabla_X W) - \nabla_X(\nabla_{[Y, Z]}W) + \nabla_{[Y, Z]}(\nabla_X W) \\ &= [\nabla_X, [\nabla_Y, \nabla_Z]](W) + [\nabla_{[Y, Z]}, \nabla_X](W) \\ &= [\nabla_X, [\nabla_Y, \nabla_Z]](W) - R(X, [Y, Z])W, \end{aligned}$$

assuming (as we may)  $\nabla_E W(p) = 0$  for any vector field  $E$ . In fact we may also assume all Lie brackets (such as  $[Y, Z]$ ) also vanish at  $p$ , and then the claim follows from the Jacobi identity (as a formal algebraic identity for commutators of operators.)

*Contractions.* In classical tensor notation:

$$\nabla_m R_{ijk}{}^l + \nabla_j R_{mik}{}^l + \nabla_i R_{jmk}{}^l = 0,$$

and contracting on  $i, l$  we find for the Ricci tensor:

$$\nabla_m R_{jk} - \nabla_j R_{mk} + \nabla_l R_{jmk}{}^l = 0.$$

Contracting again on  $j, k$ , we find:

$$\nabla^k R_{mk} - \frac{1}{2} \nabla_m R = 0,$$

the ‘twice contracted second Bianchi identity’.

## 2. Diffeomorphism invariance.

*Vector fields.* Let  $\varphi \in \text{Diff}(M)$ . The ‘push-forward’ of a vector field  $X$  and the ‘pullback’ of a function  $f$  are defined by:

$$\varphi_* X(p) := D\varphi(\varphi^{-1}p)X(\varphi^{-1}p), \quad \varphi^* f := f \circ \varphi.$$

Note that:

$$\varphi_*(fX) = (f \circ \varphi^{-1})\varphi_* X.$$

*Exercise.* As a differential operator:

$$(\varphi_* X)f|_p = X(f \circ \varphi)|_{\varphi^{-1}p},$$

or equivalently, if  $\psi = \varphi^{-1}$ :

$$\varphi^*(Xf) = (\psi_* X)(\varphi^* f).$$

*Tensors.* The pull-back of a  $(2, 0)$  tensor  $g$  is the  $(2, 0)$  tensor:

$$(\varphi^* g)(X, Y) = g(\varphi_* X, \varphi_* Y) \circ \varphi.$$

*Exercise.*  $\varphi^* g(fX, Y) := f\varphi^* g(X, Y)$ .

Similarly, for a  $(3, 1)$  tensor  $Rm$ :

$$(\varphi^* Rm)(X, Y)Z := \psi_*(Rm(\varphi_* X, \varphi_* Y)\varphi_* Z).$$

*Lie bracket.* For vector fields, there are two equivalent definitions. Let  $\varphi_t^X$  be the flow of a vector field  $X$  (one-parameter group of diffeomorphisms s.t.  $\frac{d}{dt}\varphi_t = X$ .) Given a second vector field  $Y$ , we define:

$$L_X Y := \lim_{t \rightarrow 0} \frac{1}{t}(X - \varphi_{t*} X).$$

And as a differential operator:

$$[X, Y]f := X(Yf) - Y(Xf).$$

*Exercise.*  $[X, Y] = L_X Y$ . (Do it in local coordinates.)

Clearly  $\frac{d}{dt}|_{t=0}(\varphi_{t*} Y) = -L_X Y$ . Note also that, for a function  $f$  on  $M$ :

$$\frac{d}{dt}|_{t=0}(\varphi_t^* f) = \frac{d}{dt}|_{t=0}(f \circ \varphi_t) = X(f).$$

*Exercise.*  $\varphi_{t*}[Y, Z] = [\varphi_{t*} Y, \varphi_{t*} Z]$ ; differentiating at  $t = 0$ , we recover the Jacobi identity.

The operator  $L_X$  is extended to the tensor algebra as a derivation (assuming it acts on functions as the differential operator  $X$ .) For example, for a  $(2, 0)$  tensor  $g$ :

$$(L_X g)(Y, Z) := X(g(Y, Z)) - g(L_X Y, Z) - g(Y, L_X Z).$$

*Exercise:*

$$\frac{d}{dt}(\varphi_t^* g)|_{t=0} = (L_X g)(Y, Z) = g(\nabla_Y X, Z) + g(Y, \nabla_Z X),$$

if  $\nabla$  is the Levi-Civita connection of  $g$ .

*Proposition.* Let  $\nabla^g$  be the Levi-Civita connection of a Riemannian manifold  $M$ . Define the differential operator on vector fields on  $M$ :

$$\nabla_X^* Y := \psi_*(\nabla_{\varphi_* X}^g \varphi_* Y).$$

Then  $\nabla^*$  is a connection on  $M$ , and is the Levi-Civita connection of  $\varphi^* g$ .

*Proof: guided exercise.* There are four properties to verify, and none of them is completely obvious.

- (i)  $\nabla_{fX}^* Y = f \nabla_X^* Y$ ;
- (ii)  $\nabla_X^*(fY) = (Xf)Y + f \nabla_X^* Y$ .
- (iii)  $\nabla_X^* Y - \nabla_Y^* X = [X, Y]$ .
- (iv)  $X((\varphi^* g)(Y, Z)) = (\varphi^* g)(\nabla_X^* Y, Z) + (\varphi^* g)(Y, \nabla_X^* Z)$ .

*Question:* Do you know a good ‘synthetic description’ of the Levi-Civita connection?

*Corollary.* The Riemann tensor (as an operator on the metric) is diffeomorphism-covariant:

$$Rm^{\varphi^* g} = \varphi^*(Rm^g).$$

*Exercise:* Show how this follows from the proposition, and also that it implies similar properties for the contractions  $Rc$  (Ricci curvature) and  $R$  (scalar curvature), defined by:

$$Rc^g(X, Y) := \sum_i g(Rm^g(X, E_i)E_i, Y)$$

(where  $g(E_i, E_j) = \delta_{ij}$ ) and:

$$R^g := \sum_i Rc^g(E_i, E_i).$$

### 3. Diffeomorphism invariance and the Bianchi identities.

Recall the linearization of the Ricci tensor:

$$-2(DRc)(g)[h]_{jk} = g^{pq}\nabla_p\nabla_q h_{jk} - g^{pq}(\nabla_q\nabla_j h_{kp} + \nabla_q\nabla_k h_{jp} - \nabla_j\nabla_k h_{pq}).$$

The linearization of scalar curvature is obtained from this by observing that:

$$DR(g)[h] = D(g^{jk})[h]Rc_{jk} + g^{jk}(DRc)(g)[h]_{jk}.$$

And we find:

$$-2DR(g)[h] = 2\Delta(tr_g h) - 2g^{pq}\nabla_q\nabla^j h_{jp} + 2\langle Rc, h \rangle_g.$$

The infinitesimal version of the diffeomorphism invariance of  $R$  is:

$$DR(g)[L_X g] = X(R^g).$$

*Claim.*  $DR(g)[L_X g] = 2(\nabla^k R_{kl})X^l.$

The claim implies:

$$2(\nabla^j R_{ij})X^i = X(R) = X^i\nabla_i R$$

for any  $X$ , and thus:  $\nabla^j R_{ij} - (1/2)\nabla_i R = 0$ , the twice-contracted second Bianchi identity.

**Conclusion.** The diffeomorphism invariance of scalar curvature implies the twice-contracted Bianchi II. (In fact, the diffeomorphism invariance of the Riemann tensor implies Bianchi I and Bianchi II.) And, conversely, one can use Bianchi II to recover the invariance of  $Rc$  in infinitesimal form ([Chow-Knopf p.76]).

*Proof of claim.*

Recall  $(L_X g)_{ij} = \nabla_i X_j + \nabla_j X_i$ . Using the expression above for the linearization of  $R$ , we find:

$$DR(g)[\nabla_i X_j + \nabla_j X_i] = -2g^{jk} \nabla_j \nabla_k \nabla_p X^p + g^{jk} \nabla_q \nabla_j \nabla_k X^q + g^{pq} \nabla_q \nabla_j \nabla_p X^j - 2R_{jp} \nabla^j X^p = \dots$$

In the second term on the right, commute the covariant derivatives  $\nabla_q$  and  $\nabla_j$ :

$$g^{jk} \nabla_q \nabla_j \nabla_q X^q = g^{jk} (\nabla_j \nabla_q \nabla_k X^q + R_{qjk}^l \nabla_l X^q + R_{qj}{}^q{}_l \nabla_k X^l).$$

After cancelation of curvature terms:

$$\dots = -2g^{jk} \nabla_j \nabla_k \nabla_p X^p + 2g^{jk} \nabla_j \nabla_q \nabla_k X^q - 2R_{jp} \nabla^j X^p = \dots$$

and now we commute  $\nabla_q$  and  $\nabla_k$  in the second term, which becomes:

$$2g^{jk} \nabla_j \nabla_q \nabla_k X^q = 2g^{jk} \nabla_j \nabla_k \nabla_q X^q + 2g^{jk} \nabla_j (R_{qk}{}^q{}_l X^l),$$

and continuing the computation (after cancelation of the first two terms):

$$\dots = 2g^{jk} (\nabla_j R_{kl}) X^l + 2g^{jk} R_{kl} \nabla_j X^l - 2R_{jp} \nabla^j X^p = 2(\nabla^k R_{kl}) X^l,$$

as claimed.

**4. The symbol of linearized Ricci.** Consider the linear second-order differential operator on symmetric  $(2, 0)$  tensors on  $M$ :

$$\mathcal{R} : S_M^2 \rightarrow S_M^2, \quad \mathcal{R}[h] := -2D(Rc)(g)[h]$$

$$\mathcal{R}[h] = g^{pq} (\nabla_p \nabla_q h_{jk} - \nabla_q \nabla_j h_{kp} - \nabla_q \nabla_k h_{jp} + \nabla_j \nabla_k h_{pq}).$$

The principal symbol of  $\mathcal{R}$  at a 1-form  $\zeta \in \Omega_M^1$  is the linear map (linear over functions):

$$\sigma(\mathcal{R})_\zeta : S_M^2 \rightarrow S_M^2,$$

$$\sigma(\mathcal{R})_\zeta[h]_{jk} = |\zeta|^2 h_{jk} - \zeta^p \zeta_j h_{kp} - \zeta^p \zeta_k h_{jp} + \zeta_j \zeta_k \text{tr}_g h.$$

If this map were an isomorphism for any  $\zeta \neq 0$ , the differential operator  $\mathcal{R}$  would be elliptic (by definition of ‘elliptic’!) Unfortunately this is not the case- it has an  $n$ -dimensional kernel. First we exhibit an  $n$ -dimensional subspace of  $S_M^2$  annihilated by the principal symbol of  $\mathcal{R}$ . Consider the first-order differential operator:

$$\partial : \Omega_M^1 \rightarrow S_M^2, \quad (\partial X)_{ij} = \nabla_j X_i + \nabla_i X_j.$$

(Thus  $\partial X = L_{X^*}g$ , the Lie derivative of  $g$  with respect to the dual vector field.) Its (principal) symbol is (at  $\zeta \in \Omega_M^1$ ):

$$\sigma(\partial)_\zeta : \Omega_M^1 \rightarrow S_M^2, \quad X \mapsto h_{ij} = \zeta_i X_j + \zeta_j X_i.$$

*Claim 1.* For any  $\zeta \in \Omega_M^1$  the composition:

$$\sigma(\partial)_\zeta \circ \sigma(\mathcal{R})_\zeta : \Omega_M^1 \rightarrow S_M^2$$

is the zero map. Thus:

$$Im[\sigma(\partial)_\zeta] \subset Ker[\sigma(\mathcal{R})_\zeta],$$

so the kernel has dimension at least  $n$ .

*Pedestrian proof.* Straightforward computation.

*Fancy proof.* It follows from the invariance of  $Rc$  under diffeomorphisms (in infinitesimal form) that the composition of  $\mathcal{R}$  and  $\partial$  (a priori a third-order differential operator) is in fact a first-order operator (on  $X \in \Omega_M^1$ ):

$$\mathcal{R}(\partial X) = -2DRc^g[\partial X] = -2DRc^g[L_{X^*}g] = -2L_{X^*}(Rc^g).$$

Thus the (third-order) principal symbol of the composition (which by general ‘symbol calculus’ is the composition of the principal symbols) must vanish.

Next we see the dimension of the kernel is exactly  $n$ . Inspired by the ‘twice-contracted Bianchi II’, we define a third linear differential operator (of order 1):

$$B = B^g : S_M^2 \rightarrow \Omega_M^1, \quad B(h)_k = g^{ij}(\nabla_i h_{jk} - \frac{1}{2}\nabla_k h_{ij}),$$

with principal symbol (at  $\zeta \in \Omega_M^1$ ):

$$\sigma(B)_\zeta : S_M^2 \rightarrow \Omega_M^1, \quad h \mapsto g^{ij}(\zeta_i h_{jk} - \frac{1}{2}\zeta_k h_{ij}) = \zeta^j h_{jk} - \frac{1}{2}\zeta_k tr_g h.$$

*Claim 2.* For any  $\zeta \in \Omega_M^1$ , the composition of symbols:

$$\sigma(B)_\zeta \circ \sigma(\mathcal{R})_\zeta : S_M^2 \rightarrow \Omega_M^1$$

is the zero map. Thus:

$$Im[\sigma(\mathcal{R})_\zeta] \subset K_\zeta := Ker[\sigma(B)_\zeta].$$

*Pedestrian proof.* Straightforward computation.

*Fancy proof.* The twice-contracted Bianchi II says exactly that  $B^g(Rc^g) = 0$ . To linearize this identity with respect to  $g$ , we write  $B^g(h)$  as the function of two arguments  $B(g, h)$  (linear in the second argument) and differentiate the composite map  $B(g, Ric^g)$  as a function of  $g$  in the direction of  $h \in S_M^2$  (using the chain rule):

$$D_1B(g, Rc^g)[h] + D_2B(g, Rc^g)[D(Rc^g)h] = 0.$$

By linearity of  $B$  in its second argument,  $D_2B(g, Rc^g)[v] = B(g, v)$  (for all  $v \in S_M^2$ ), and  $D_1B$  is easily computed:

$$D_1B(g, q)[h] = -g^{ip}g^{qj}h_{pq}(\nabla_i q_{jk} - \frac{1}{2}\nabla_k q_{ij}), \quad q \in S_M^2, h \in S_M^2.$$

So the Bianchi identity implies  $D_1B(g, Rc^g)[h] = 0$  for all  $h$ , and again by linearity we conclude:

$$B(g, -2D(Rc^g)[h]) = 0 \quad \forall h \in S_M^2,$$

which means  $B^g \circ \mathcal{R}$  is the zero operator  $\Omega_M^1 \rightarrow S_M^2$ .

We are now interested in the adjoint of the linear map  $\sigma(B)_\zeta$  (where  $\Omega_M^1$  and  $S_M^2$  have the inner products naturally induced by  $g$ .) It would be nice if this were just the already defined  $\sigma(\partial)_\zeta$ , but since we subtract the trace in the definition of  $B^g$ , this is not true. Instead, consider:

$$\beta_\zeta : \Omega_M^1 \rightarrow S_M^2 \quad X \mapsto h_{ij} = \frac{1}{2}(\zeta_i X_j + \zeta_j X_i - \langle \zeta, X \rangle_g g_{ij}).$$

*Claim 3.*  $\langle \beta_\zeta[X], h \rangle_g = \langle X, \sigma(B_\zeta)[h] \rangle_g$ , for all  $\zeta, X, h$ .

*Proof.* Straightforward computation.

Thus  $\beta_\zeta$  is the adjoint of  $\sigma(B)_\zeta$ , and denoting by  $K_\zeta \subset S_M^2$  the kernel of the latter map, it follows that:

$$Im(\beta_\zeta) = K_\zeta^\perp.$$

*Exercise.* Show that, like  $\sigma(\partial)_\zeta$ , the map  $\beta_\zeta$  has trivial kernel, and therefore  $n$ -dimensional range. This implies  $dim(K_\zeta) = \frac{n(n+1)}{2} - n = \frac{n(n-1)}{2}$ .

It is now very easy to check that  $\sigma(\mathcal{R}_\zeta)$  not only leaves the subspace  $K_\zeta$  invariant, but in fact acts on it like  $|\zeta|^2$  times the identity! Indeed,  $h \in K_\zeta$  if, and only if,  $\zeta^j h_{jk} = (1/2)\zeta_k tr_g h$ , so:

$$\sigma(\mathcal{R})_\zeta[h]_{jk} = |\zeta|^2 h_{jk} - \frac{1}{2}\zeta_j \zeta_k tr_g h - \frac{1}{2}\zeta_k \zeta_j tr_g h + \zeta_j \zeta_k tr_g h = |\zeta|^2 h_{jk}.$$

Thus the dimension of  $Ker[\sigma(\mathcal{R})_\zeta]$  is at most  $n$ , and therefore exactly  $n$ . Therefore this kernel coincides with the image of  $\sigma(\partial_\zeta)$ .

*Remark.* It should be clear from the above discussion that the linear map  $\sigma(\mathcal{R})_\zeta$  is not self-adjoint on  $S_M^2$ .