I. INTRODUCTION TO MEAN CURVATURE FLOW

1. The Hessian and the second fundamental form.

Let M be a manifold endowed with a torsion-free connection ∇. The *Hessian* of $f \in C^2(M)$ is the symmetric 2-tensor:

$$\nabla^2 f(X,Y) := X(dfY) - df(\nabla_X Y), \quad X, Y \in TM.$$

If ∇ is the Levi-Civita connection of a metric g on M, we obtain the *Laplace-Beltrami operator* Δ_g by taking the trace:

$$\Delta_g f = tr_g \nabla^2 f.$$

If M is a submanifold of $\mathbb{R}^p$, the induced connection ∇ and the (vector) second fundamental form $\vec{A}$ are defined as the tangential (resp. normal) component of the euclidean directional derivative:

$$X(Y) = \nabla_X Y + \vec{A}(X,Y) \in TM \oplus TM^{\perp}, \quad X, Y \in TM,$$

where on the left-hand side Y is regarded as a function on M, with values in $\mathbb{R}^p$.

A diffeomorphism of manifolds $F : M_0 \to M$ induces in a natural way a 'push-forward' map $X \mapsto X^F$ from vector fields on M_0 to vector fields on M:

$$X^F(q) = dF(F^{-1}q)X(F^{-1}q), \quad q \in M;$$

equivalently, as a differential operator on functions on M: $X^F\varphi = X(\varphi \circ F^{-1})$. Thus, given a connection ∇ on M, F induces a 'pullback connection' ∇^* on M_0 via:

$$(\nabla^*_X Y)^F = \nabla_{X^F} Y^F.$$

If ∇ is torsion-free, so is ∇^* (since $[X^F, Y^F] = [X,Y]^F$); if ∇ is compatible with a metric g on M, ∇^* is compatible with the 'pullback metric' F^*g (since $X\langle Y, Z\rangle_{F^*g} = (X^F\langle Y^F, Z^F\rangle_g) \circ F$.)

Combining these definitions, the following proposition has a one-line proof:

Proposition. Let $F : M_0 \to \mathbb{R}^p$ induce a diffeomorphism from M_0 onto a submanifold $M \subset \mathbb{R}^p$ (endowed with the connection induced from $\mathbb{R}^p$). Then the Hessian of F (regarded as a vector-valued function on M_0) with respect to the pull-back connection ∇^* equals the pull-back of the second fundamental form of M:

$$\nabla^{*2}F(X,Y)|_p = \vec{A}(X^F, Y^F)|_{F(p)}.$$

Proof:

$$X(dFY) - dF(\nabla^*_X Y)|_p = X^F(Y^F) - \nabla_{X^F}Y^F|_{F(p)} = \vec{A}(X^F, Y^F)|_{F(p)}.$$

In particular, if g is the induced metric on M and Δ^* is the Laplacian on M_0 for the pullback metric F^*g, we have:

$$\Delta^* F|_p = \vec{H}|_{F(p)},$$

the mean curvature vector of M.

2. First variation of area. Let M^n be an orientable compact manifold (possibly with boundary) endowed with a volume n-form $\omega \in \Omega^n(M)$. The 'area' of M is:

$$A(M) = \int_M \omega.$$

If φ is an orientation-preserving diffeomorphism of M, its *Jacobian* is the positive function J_φ on M defined by $\varphi^*\omega = J_\varphi \omega$. A Riemannian metric g on M defines a volume form ω_g by the requirement:

$$\omega_g(e_1, \dots e_n) = 1 \text{ if } (e_i) \text{ is } g-\text{orthonormal}.$$

The following is a standard formula:

Lemma. If $\omega = \omega_g$ and $\varphi \in Diff^+(M)$, for any $p \in M$ and any orthonormal frame (e_i) at p, we have:

$$J_\varphi(p) = \sqrt{det(g_{ij})},$$

where $g_{ij} = \langle d\varphi e_i, d\varphi e_j \rangle_g$.

Now let $M_0 \subset \mathbb{R}^{n+1}$ be a hypersurface, $F^\epsilon : M_0 \to \mathbb{R}^{n+1}$ a family of embeddings, with $F^0 = Id$, $M_\epsilon = F^\epsilon(M_0)$. The area of M_ϵ is:

$$A(M_\epsilon) = \int_{M_0} \omega_\epsilon,$$

where ω_ϵ is the volume form defined by the pullback metric $F^{\epsilon *}g_{eucl}$. Let $V : M_0 \to \mathbb{R}^{n+1}$ be the 'variational vector field':

$$V = \frac{d}{d\epsilon}|_{\epsilon=0} F^\epsilon.$$

The first variation of area is given by the following formula.

Proposition (First variation of area). If M_0 is compact without boundary:

$$\frac{d}{d\epsilon}|_{\epsilon=0}A(M_\epsilon) = -\int_{M_0}\langle\vec{H}, V\rangle\omega_0.$$

If M_0 has boundary ∂M_0:

$$\frac{d}{d\epsilon}|_{\epsilon=0}A(M_\epsilon) = -\int_{M_0}\langle\vec{H}, V\rangle\omega_0 + \int_{\partial M_0}\langle V, n\rangle\sigma_0,$$

where $n \in TM_0$ is a choice of unit normal vector field for ∂M_0 and $\sigma_0 = i_n\omega_0 \in \Omega^{n-1}(\partial M_0)$ is the induced boundary volume form.

Proof. Let ω_0 be the volume form for the induced metric g_0 on M_0. For the Jacobian of F^ϵ, we have:

$$\omega_\epsilon = J_\epsilon\omega_0 \Rightarrow J_\epsilon = \sqrt{\det(g_{ij}^\epsilon)},$$

where

$$g_{ij}^\epsilon = \langle dF^\epsilon e_i, dF^\epsilon e_j\rangle_{g_0} = \langle e_i(F^\epsilon), e_j(F^\epsilon)\rangle_{g_0},$$

(e_i) a local orthonormal frame (w.r.t. g_0).

Fix $p \in M_0$. Since $J_0 \equiv 1$, we have:

$$\frac{d}{d\epsilon}|_{\epsilon=0}J_\epsilon(p) = \frac{1}{2}\frac{d}{d\epsilon}|_{\epsilon=0}\log\det(g_{ij}^\epsilon) = \frac{1}{2}g_0^{ij}(\frac{d}{d\epsilon}|_{\epsilon=0})g_{ij}^\epsilon,$$

and:

$$\frac{1}{2}g_0^{ij}(\frac{d}{d\epsilon}|_{\epsilon=0})g_{ij}^\epsilon = \frac{1}{2}g^{ij}(\langle e_i(V), e_j\rangle + \langle e_i, e_j(V)\rangle).$$

We consider two cases:

(i) V is a normal vector field: $V = \varphi N$, for some $\varphi \in C^1(M_0)$. Then (with inner products w.r.t. g_0):

$$\langle e_i(V), e_j\rangle = \varphi\langle e_i(N), e_j\rangle = -\varphi\langle N, e_i(e_j)\rangle = -\varphi A(e_i, e_j),$$

and taking trace w.r.t. g_0:

$$\frac{d}{d\epsilon}|_{\epsilon=0}J_\epsilon(p) = -\varphi(p)H(p) = -\langle\vec{H}, V\rangle(p).$$

(ii) $V : M_0 \to \mathbb{R}^{n+1}$ is a tangent vector field, $V \in TM_0$. Then $e_i(V) = \nabla_{e_i} V + A(e_i, V)N$, so:

$$\frac{d}{d\epsilon}|_{\epsilon=0} J_\epsilon(p) = \sum_i \langle e_i(V), e_i \rangle = \sum_i \langle \nabla_{e_i} V, e_i \rangle := div^{M_0} V(p),$$

the divergence of V at p. Now we just have to observe that the divergence has the property:

Lemma. If $\partial M_0 = \emptyset$, the integral of $div^{M_0} V$ over M_0 is zero (for any vector field V); if $\partial M_0 \neq \emptyset$:

$$\int_{M_0} (div^{M_0} V)\omega_0 = \int_{\partial M_0} \langle V, n \rangle \sigma_0.$$

Proof. Denote by $\mathcal{L}_V$ the Lie derivative on M_0 with respect to V. We have the standard relation:

$$\mathcal{L}_V \omega_0 = d(i_V \omega_0),$$

and also:

$$\mathcal{L}_V \omega_0 = (div^{M_0} V)\omega_0.$$

(*Exercise*). So the lemma follows from Stokes' theorem.

Given the lemma, the proof of the proposition is completed by writing an arbitrary $V \in C^1(M_0, \mathbb{R}^{n+1})$ as the sum of its tangential and normal components.

3. Mean curvature flow and mean curvature motions.

Mean curvature flow(MCF) is a flow in the space of embeddings (from a fixed n-manifold M_0 to $\mathbb{R}^{n+1}$):

$$F : M_0 \times [0, T) \to \mathbb{R}^{n+1},$$

defined by:

$$\partial_t F(x, t) = \vec{H}(F(x, t)).$$

From section 1, if we endow M_0 with the pullback metric $g(t) = F(t)^* g_{eucl}$, we can write this as a 'heat equation' for the corresponding Laplace-Beltrami operator $\Delta_{g(t)}$ on M_0:

$$\partial_t F = \Delta_{g(t)} F.$$

Now consider what happens if we compose a mean curvature flow with a time-dependent family of diffeomorphisms $\varphi(t) \in Diff(M_0)$ with generator $X(t) = \partial_t \varphi \in TM_t$ ($M_t = F(t)(M_0)$ is the hypersurface at time t). This gives a new family of embeddings:

$$G(x,t) = F(\varphi(x,t),t),$$

$$\partial_t G = DF(t) \cdot \partial_t \varphi + \partial_t F(\varphi(x,t),t) = DF(t) \cdot X + \vec{H}(G(x,t)),$$

so $G(t)$ is not a mean curvature flow, but parametrizes the same family of hypersurfaces, $G(t)(M_0) = M_t$. On the other hand, for the normal component of the velocity vector we still have:

$$\langle G(x,t), N(G(x,t)) \rangle = \langle \partial_t F(\varphi(x,t),t), N(G(x,t) \rangle = H(G(x,t)).$$

In most cases we are interested in the evolving hypersurfaces M_t, so it makes sense to consider 'mean curvature flows modulo reparametrizations' (that is, a flow in the 'space of hypersurfaces diffeomorphic to M_0'). So we define a *mean curvature motion* to be a one-parameter family of hypersurfaces M_t in $\mathbb{R}^{n+1}$ which admits a parametrization $F(t) : M_0 \to M_t \subset \mathbb{R}^{n+1}$ over a fixed manifold, with normal velocity equal to mean curvature:

$$\langle \partial_t F, N \rangle = H \circ F(t).$$

Proposition. If $F(t)$ parametrizes a mean curvature motion, one may find a curve $\varphi(t) \in Diff(M_0)$ so that $G(x,t) = F(\varphi(x,t),t)$ is a mean curvature flow.

Proof. [K.Ecker] Let $\varphi(x,t)$ be the solution of:

$$\partial_t \varphi(x,t) = -[DF(\varphi(x,t),t)]^{-1} (\partial_t F(\varphi(x,t),t))^{TM_t},$$

where the superscript denotes 'tangential component'. This is regarded as a system of (non-autonomous) ODE on M_0; by standard existence theory, if M_0 is compact, there is indeed a family $\varphi(t)$ solving this system, defined for all t for which $F(t)$ is defined. Then if $X = \partial_t \varphi$ is the generator and $G(x,t)$ is defined as above, we have:

$$\partial_t G(x,t) = DF(\varphi(x,t),t) X(x,t) + \partial_t F(\varphi(x,t),t) = [\partial_t F(\varphi(x,t),t]^{Nor} = \vec{H}(G(x,t)),$$

(since F parametrizes a MCM). So G parametrizes a mean curvature flow.

MCF/MCM as a gradient flow. From the first variation formula, we have for a family of embeddings $F(t) : M_0 \to \mathbb{R}^{n+1}$ (if M_0 is compact without boundary):

$$\frac{d}{dt}Area(M_t) = -\int_{M_t} \langle \vec{H}, \partial_t F\rangle d\omega_t = -\int_{M_t} H\langle \partial_t F, N\rangle d\omega_t$$

(where ω_t is the volume element for the metric $g(t)$ on M_t induced from the euclidean metric). So we see that (MCF) is the (negative) gradient flow of area in the space of embeddings, while (MCM) is the (negative) gradient flow of area in the space of hypersurfaces. In particular, any solution of (MCM) (in particular, of (MCF)) satisfies:

$$\frac{d}{dt}Area(M_t) = -\int_{M_t} H^2 d\omega_t.$$

Another useful way to arrive at this result is to write:

$$Area(M_t) = \int_{M_0} J_t\omega_0, \text{ where } F(t)^*\omega_t = J_t\omega_0,$$

and then use:

$$\frac{d}{dt}J_t = -\langle \vec{H}, \partial_t F\rangle = -H^2,$$

for any mean curvature motion.

Quasilinear parabolic equations on manifolds. The operator defining MCF depends on the induced connection and on the induced metric on M_0, both time-dependent:

$$\partial_t F = \vec{H} \circ F = tr_{g(t)}(\nabla^t)^2 F,$$

where ∇^t is the pullback under $F(t)$ of the induced connection on M_t and $g(t) = F(t)^* g_{eucl}$.

If ∇^0, ∇^1 are torsion-free connections on M_0, the difference $\Gamma = \nabla^1 - \nabla^0$ is a symmetric $(2,1)$ tensor on M_0:

$$\Gamma(X,Y) = \nabla^1_X Y - \nabla^0_X Y = \Gamma(Y,X)$$

(check these claims!) For the associated Hessians of a C^2 function $G : M_0 \to \mathbb{R}^p$, we have:

$$(\nabla^1)^2 G(X,Y) - (\nabla^0)^2 G(X,Y) = DG(\nabla^0_X Y - \nabla^1_X Y) = -DG(\Gamma(X,Y)).$$

Thus we are led to consider a flow of hypersurface embeddings $G(t) : M_0 \to \mathbb{R}^{n+1}$ satisfying the equation:

$$\partial_t G = tr_{g(t)}(\nabla^0)^2 G, \quad g(t) = G(t)^* g_{eucl},$$

for a fixed connection ∇^0 on M_0. (The point is that now the connection is time-independent). It is easy to see that this defines a mean curvature motion: if ∇^t is the pullback under $G(t)$ of the induced connection on $M_t = G(t)M_0$ and $\Gamma^t = \nabla^t - \nabla^0$, we see that $DG(t)(\Gamma^t(X,Y))$ is tangent to M_t, and hence:

$$H(G(x,t)) := \langle tr_{g(t)}(\nabla^t)^2 G, N(G(x,t)) = \langle tr_{g(t)}(\nabla^0)^2 G, N(G)(x,t)\rangle.$$

($N(G)$ can be thought of as a fixed nonlinear function of DG).

In particular $G(t)$ can be reparametrized to a solution of (MCF). The advantage of writing the equation with a fixed connection is the explicit 'quasilinear parabolic structure'. To make sense of what this means on a manifold, we need a fixed (but arbitrary) connection. We have the following definition.

Definition. A *quasilinear second order parabolic system* on a manifold M for an unknown function $u : M \times [0,T) \to \mathbb{R}^p$ is a system of PDE on M of the form:

$$\partial_t u - tr_g \nabla^2 u + f(x,t,u,Du) = 0,$$

where ∇ is a fixed torsion-free connection on M and $g(x,t,u,Du)$ is a Riemannian metric on M (depending on the indicated quantities).

Note that the definition is independent of the connection (the difference of the Hessians for two connections is a first-order operator); so we can take, for example, the Levi-Civita connection for an arbitrary fixed metric on M.

This will be useful for local existence: we can appeal to the methods of quasilinear parabolic theory on $\mathbb{R}^n$, applied to the equation satisfied by G above.

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