TOPOLOGY PRELIM REVIEW 2021: LIST TWO

Topic: separation properties and existence of continuous functions

1. A Hausdorff space X is regular if, and only if:

$$(\forall x \in X)(\forall U_x)(\exists V_x \subset U_x \text{ open })(\overline{V_x} \subset U_x).$$

2. A space X is *completely regular* if for any $p \in X$ and any $C \subset X$ closed such that $p \notin C$, there exists $f : X \to [0,1]$ continuous with $f(p) = 0, f \equiv 1$ on C.

(i) Show directly that metric spaces are completely regular (i.e., find an explicit f).

(ii) Show that normal Hausdorff spaces are completely regular. (This gives another proof of (i)).

3*. A Hausdorff space X is *completely normal* if, given any two separated sets F_1, F_2 of X (meaning: $\overline{F_1} \cap F_2 = \emptyset = F_1 \cap \overline{F_2}$)) one may find disjoint open neighborhoods U_1, U_2 of F_1, F_2 (resp.) Clearly completely normal spaces are normal.

Prove that metric spaces are completely normal.

4*. Any Hausdorff, regular, second countable space is completely normal. (Prove directly, without using Urysohn metrization.)

Hint: Recall the proof of the fact that regular, second countable spaces are normal.

5. A Hausdorff space X is normal iff for any $A \subset X$ closed, and any $U \supset A$ open, we may find $V \subset X$ open, so that

$$A \subset V \subset \overline{V} \subset U.$$

6. If X, Y are metric spaces, Y is complete, $A \subset X$ and $f : A \to Y$ is uniformly continuous on A, then there is a unique extension of f to a continuous map $F : \overline{A} \to Y$.

Hint: Existence has two parts: defining the map and proving it is continuous. Uniqueness is easy.

7. Let X be a normal space, $C \subset X$ closed, $f : C \to \mathbb{R}$ continuous. Use Tietze's extension theorem and a homeomorphism from $\mathbb{R}$ to $(-1,1)$ to show f admits a continuous extension $F : X \to \mathbb{R}$.

8. Any closed subset of a metric space is a G_δ (use the distance function.)

9. Let X be normal, $A \subset X$ non-empty. There exists $f \in C(X;[0,1])$ with $\{x \in X; f(x) = 0\} = A$ iff A is a closed G_δ subset of X.

Hint. One direction is easy. For the other, suppose $A = \bigcap_{n \geq 1} U_n$, where we may assume (show this) $U_n \supset U_{n+1}$. Let $f_n \in C(X;[0,1])$ be a solution to the

Urysohn problem for the pair of closed sets (A, U_n^c), with $f_n \equiv 0$ in A, $f_n \equiv 1$ in U_n^c. Now consider:

$$f : X \to [0,1], \quad f(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} f_n(x).$$

Show that f is continuous with values in $[0,1]$, and that $x \notin A \Rightarrow f(x) > 0$. Note that if $B \subset X$ is closed and disjoint from A, f can be chosen to satisfy $f \equiv 1$ in B (just assume $B \subset U_n^c$ for all n.)

10. *(Strong Urysohn Lemma.)* Let X be normal, $A, B \subset X$ closed disjoint. Show there exists $f \in C(X;[0,1])$ so that $f^{-1}(0) = A, f^{-1}(1) = B$ iff A and B are G_δ sets.

Hint: If A, B are closed G_δ sets, let $f_A, f_B \in C(X,[0,1])$ be as in the previous problem:

$$f_A^{-1}(0) = 1, \quad f_A \equiv 0 \text{ in } B; \quad f_B^{-1}(0) = B, \quad f_B \equiv 0 \text{ in } A.$$

Then consider:

$$f(x) = \frac{f_A(x)}{f_A(x) + f_B(x)}.$$

11. A space X is *perfectly normal* if every closed subset is a G_δ (example: metric spaces.) Prove that any perfectly normal space is completely normal (see problem 3 for the def.) *Hint:* Given two separated sets A, B consider f, g (cont.) from X to $[0,1]$ (resp.) vanishing only on $\overline{A}, \overline{B}$ (resp.) Then consider the sets $\{x; f(x) < g(x)\}$ and $\{x; f(x) > g(x)\}$.