REVIEW LIST 7: DIFFERENTIAL TOPOLOGY (version date: 6/22)

1. (i) Give an example of an injective immersion of manifolds that is not an embedding.

(ii) Any smooth immersion $f : X \to Y$ is locally an embedding, in the following sense: for any $p \in X$, there exists an open neighborhood $U \subset X$ of p such that the restriction $f_{|U} : U \to Y$ is an embedding.

(iii) Show that an injective smooth immersion of a compact manifold is an embedding.

2. Let $f : S^1 \to \mathbb{R}$ be a C^1 map, $y \in \mathbb{R}$ a regular value, Prove that $f^{-1}(y)$ has an even number of points.

3. (i) Let $f : R \to R$ be a local diffeomorphism. Prove that the image of f is an open interval, and that f maps R diffeomorphically onto this interval.

(ii) Find a local diffeomorphism $f : R^2 \to R^2$ which is not a diffeomorphism onto its image.

(iii) Prove that an *injective* local diffeomorphism $f : X \to Y$ is a diffeomorphism from X to an open subset of Y.

4. (i) If $f : X \to Y$ is a submersion, then f is an open map.

(ii) If X is compact and Y is connected, every submersion $f : X \to Y$ is surjective.

(iii) There exist no submersions from compact manifolds to euclidean spaces.

(iv) If M is a compact n-dimensional manifold and $f : M \to R^n$ is a smooth map, f cannot be an immersion.

5. Let M_n, S_n be the vector spaces of $n \times n$ matrices (resp. $n \times n$ symmetric matrices), and let $f : M_n \to S_n$ be the smooth map $f(A) = AA^t$ (the superscript t means 'transpose'.)

(i) Compute the differential of f at an arbitrary matrix A.

(ii) Show that the identity matrix I_n is a regular value of f, and therefore the orthogonal group $O(n)$ is a manifold (compute its dimension).

(iii) Compute the tangent spaces $T_{I_n}O(n)$ and $T_AO(n)$ for $A \in O(n)$.

(iv) Show that $O(n)$ is compact.

6. Let $p : R^k \to R$ be a homogeneous polynomial of degree d in k variables:

$$p(tx) = t^d p(x); \quad t \in R, x \in R^k.$$

(i) Prove that if $a \neq 0$ the set $M_a = \{x \in R^k; p(x) = a\}$ is a smooth hypersurface in R^k (codimension 1 submanifold). *Hint:* use the Euler identity:

$$\sum_{i=1}^{k} x_i \frac{\partial p}{\partial x_i} = dp$$

to show any $a \neq 0$ is a regular value of p.

(ii) Prove that all M_a with $a > 0$ are diffeomorphic to one another.

7. (i) Let V be a finite-dimensional real vector space, $T \in \mathcal{L}(V)$, $\Delta \subset V \times V$ the diagonal subspace, $\Gamma_T \subset V \times V$ the graph subspace of T. Then:

$$\Gamma_T \pitchfork \Delta \Leftrightarrow 1 \text{ is not an eigenvalue of } T.$$

In this case, what is the dimension of the intersection subspace $\Gamma_T \cap \Delta$?

(ii) A smooth map $f : M \to M$ of a manifold M is a *Lefschetz map* if 1 is not an eigenvalue of $df(x) \in \mathcal{L}(T_x M)$, for any fixed point x of f. Prove that if M is a compact manifold and $f : M \to M$ is a Lefschetz map, then f has only finitely many fixed points.

8. A vector field on M can be described in two ways: (i) In local coordinates $(x, y) \in U_0 \times R^n$ on the tangent bundle TM, as a map $y = X(x), X : U_0 \to R^n$; (ii) as a section $\sigma : M \to TM$ of the tangent bundle $\pi : TM \to M$, meaning $\pi \circ \sigma = Id_M$. A singularity of σ (or X) is a point $p \in M$ such that $\sigma(p) = 0_p$, a point of the zero-section $\Sigma_0 \subset TM$ (in local coordinates, a point $x_0 \in U_0, X(x_0) = 0$). A *simple singularity* is a singularity x_0 at which $dX(x_0)$ has rank $n = dim(M)$. Show that the singularity x_0 of X is simple iff $\sigma \pitchfork_p \Sigma_0$ (where p corresponds to x_0 in coordinates.)

(Def.: $\Sigma_0 = \{0_p; p \in M\}$ is the 'zero section' of TM). Show that if $\sigma \pitchfork \Sigma_0$, the singularities of X are isolated.

Remark. Note that $d\sigma(p) \in \mathcal{L}(T_p M, T_{0_p} TM)$ always has rank n, since $d\pi(0_p) \circ d\sigma(p) = \mathbb{I}_{T_p M}$, where $\sigma(p) = 0_p$.

9. *Differentiable Urysohn lemma.* Let M be a smooth manifold, $A, B \subset M$ disjoint closed subsets. Show there exists a smooth (function. $f : M \to [0, 1]$ so that $f \equiv 0$ on A, $f \equiv 0$ on B. *Hint:* smooth partition of unity strictly subordinate to $\{A^c, B^c\}$.

10. (i) Let M be a differentiable manifold, of class C^{k+1}. Define 'Riemannian metric of class C^k' on M.

(ii) Use partitions of unity to show any differentiable manifold admits a Riemannian metric.

11. On any smooth manifold X there exists a smooth proper function $f : X \to \mathbb{R}$.

Hint: Let $\{U_\alpha\}$ be the family of all precompact open subsets of X, $(\phi_i)_{i \geq 1}$ a subordinate smooth partition of unity. Consider:

$$f(x) = \sum_{i=1}^{\infty} i \phi_i(x).$$

Show that f is well-defined, smooth and proper.

12. Let M be a 2-dimensional *compact* manifold of class C^r, which can be covered by n domains of coordinate charts $U_1, \ldots, U_n$, $h_i : U_i \to B(3)$, the open ball of radius 3 in R^2. Let $\phi \in C^\infty(R^2)$ be a smooth 'bump function': equal to

1 in $B(1)$, equal to 0 in the complement of $B(2)$. Let $\varphi_i = \phi \circ h_i$ in U_i, extended to zero outside of U_i (so $\varphi_i \in C^r(M)$.)

Consider the map $f : M \to R^{3n} = R \times \ldots \times R \times R^2 \times \ldots \times R^2$ (n factors equal to R and n factors equal to R^3):

$$f(x) = (\varphi_1(x), \ldots, \varphi_n(x), \varphi_1 h_1(x), \ldots, \varphi_n h_n(x).).$$

Then f is an injective immersion (and therefore an embedding, since M is compact.)

13. Let $f : X \to R^N$ be an injective immersion, where X is a k-dimensional manifold and $N > 2k+1$. Define the maps:

$$h : X \times X \times R \to R^N, \quad h(x, y, t) = t(f(x) - f(y)).$$

$$g : TX \to R^N, \quad g(x, v) = df(x)[v].$$

(i) Show there exists $a \in R^N$ nonzero which is neither in the image of h nor in the image of g. (*Hint:* Sard's theorem.)

(ii) Show that for such a, if $H \subset R^N$ is the orthogonal complement of the one-dimensional subspace spanned by a and $\pi : R^N \to H$ the orthogonal projection, then $\pi \circ f : X \to H$ is injective.

(iii) Show that $\pi \circ f$ is an immersion.

Conclusion: If the manifold X is compact, X can be embedded into R^{2k+1}.

14. Show that if X is a k-dimensional compact smooth manifold, there exists an immersion $f : X \to R^{2k}$.

Let X, Y be differentiable manifolds; fix a metric d on Y.

We denote by $C^0(X, Y)$ the space of continuous maps from X to Y, with the topology of uniform convergence on compact sets. Recall a basis of neighborhoods of $f \in C^0(X, Y)$ is given by the sets:

$$V(f, K, \delta) = \{g \in C^0(X, Y); d(f(x), g(x)) < \delta, \forall x \in K\}.$$

(If X is compact, this is the same as the uniform topology, with the basis $\{V(f, \delta)\}$ given by taking $K = X$ above.)

15. $C^0(X, Y)$ is metrizable. If, furthermore, the metric space Y has a countable basis, the same holds for $C^0(X, Y)$.

16. Let $g, h : M \to R$ be continuous functions on a C^k manifold M, with $h(p) < g(p), \forall p \in M$. Then there exists a C^k function $f : M \to R$, so that $h < f < g$ on M.

Hint. For each $p \in M$, let $a_p = (1/2)(g(p) + h(p))$, so $h(p) < a_p < g(p)$. Thus for some neighborhood V_p, $h(q) < a_p < g(q)$ for $q \in V_p$. This defines an open cover $\mathcal{C} = (V_p)_{p \in M}$ of M. Consider a C^k partition of unity $(\varphi_p)_{p \in M}$ subordinate to $\mathcal{C}$. Then let:

$$f = \sum_{p \in M} \varphi_p a_p.$$

(ii) Let M be a C^k manifold, $g : M \to R^n$ a continuous function. Given any positive continuous function $\epsilon : M \to R^+$, there exists a C^k function $f : M \to R^n$ so that $|f(x) - g(x)| < \epsilon(x)$ on M.

(iii) If M is a compact C^k manifold, $C^k(M, R^n)$ is dense in $C^0(M; R^n)$.

The C^1 topology. Let M, N be differentiable manifolds of class C^k $(k \geq 1)$. We assume the existence of an embedding $\Phi : N \to R^n$ of class C^k, for some n. Indeed to simplify the notation we'll just assume N is a surface oof class C^k in R^n. Fix a Riemannian metric on M, of class C^{k-1} (that is, at least of class C^0.)

On the space of C^1 maps from M to N a topology is given by C^1-uniform convergence on compact subsets of M; we'll denote this topological space by $C^1(M, N)$, The basic neighborhoods of $f \in C^1(M, N)$ are the sets $V^1(f, K, \delta)$, where $K \subset M$ is compact and δ is a positive real number:

$$V^1(f, K, \delta) = \{g \in C^1(M, N); |f(p) - g(p)| < \delta \text{ and } |df(p) - dg(p)| < \delta, \forall p \in K\}.$$

(we may abbreviate this by saying $||f - g||_{C^1(K)} < \delta$.)

When M is compact, we may take $K = M$ to define basis sets $V^1(M, \delta)$; this is the topology of C^1-uniform convergence on M.

Stability of certain classes of C^1 maps.

If a C^1 map (of differentiable manifolds) is an immersion, a submersion, an embedding, a diffeomorphism or transversal to a closed submanifold, then this property is preserved under small perturbations fo the map in the C^1 topology (if the domain manifold is compact.)

17. (i) Let $\mathcal{O} \subset \mathcal{L} = \mathcal{L}(R^m, R^n)$ denote the set of injective linear maps. Show that $\mathcal{O}$ is open $(m \leq n)$.

(ii) Let $U \subset R^m$ open, $K \subset U$ compact. Let $f \in C^1(U, R^n)$ be an immersion in K (that is, if $x \in K$, $df(x) \in \mathcal{L}(R^m, R^n)$ has trivial kernel.) Then there exists $\eta > 0$ so that if $g \in C^1(U, R^n), ||g - f||_{C^1(K)} < \eta$, then the restriction $g_{|K}$ is an immersion.

(iii) Assume M is compact. The C^1 immersions define an open subset $Imm^1(M, N) \subset C^1(M, N)$.

Hint: Let $M = \bigcup U_i$ be a finite open cover of M, with U_i the domain of a chart for M and (V_i) with $V_i \subset \overline{V_i} \subset U_i$ also a finite open cover.

18. (i) Let $\mathcal{S} \subset \mathcal{L} = \mathcal{L}(R^m, R^n)$ denote the set of surjective linear maps. Show that $\mathcal{S}$ is open $(m \geq n)$.

(ii) Let $U \subset R^m$ open, $K \subset U$ compact. Let $f \in C^1(U, R^n)$ be a submersion in K (that is, if $x \in K$, $df(x) \in \mathcal{L}(R^m, R^n)$ is surjective.) Then there exists $\eta > 0$ so that if $g \in C^1(U, R^n), ||g - f||_{C^1(K)} < \eta$, then the restriction $g_{|K}$ is a submersion.

(iii) Assume M is compact. The C^1 submersions define an open subset $Sub^1(M,N) \subset C^1(M,N)$.

19. (i) Let $U \subset R^m$ open, $K \subset U$ compact convex, $f : U \to R^n$ a C^1 map such that $f_{|K}$ is an embedding. Prove there exists $\eta > 0$ so that if $g \in C^1(U, R^n)$ with $||g - f||_{C^1(K)} < \eta$, then $g_{|K}$ is an embedding.

Hint. We know there exists $\eta' > 0$ so that $||g - f||_{C^1(K)} < \eta' \Rightarrow g_{|K}$ is an immersion. We also know there exist $c > 0, \delta > 0$ so that $|f(x) - f(y)| > c|x - y|$ for any $x \in K, y \in U$ with $|x - y| < \delta$. By compactness, there exists $d > 0$ so that $|f(x) - f(y)| > d$ if $(x, y) \in A = \{(x, y) \in K \times K; |x - y| \geq \delta\}$, a compact set.

Set $h = g - f$. Then $|h(x)| < \eta, |dh(x)| < \eta$, for all $x \in K$, By the mean value inequality (since K is convex) we have $|h(x) - h(y)| < \eta|x - y|$, for all $x, y \in K$. Let $\eta = \min\{\eta', \frac{c}{2}, \frac{d}{3}\}$ and complete the injectivity proof by considering two cases: (i) $0 < |x - y| < \delta$ and (ii) $|x - y| \geq \delta$ (then $(x, y) \in A$).

(ii) If M is compact, the C^1 embeddings $f : M \to N$ define an open subset $Emb^1(M, N) \subset C^1(M, N)$.

Hint: Let $\{W_i \subset V_i \subset \overline{V_i} \subset U_i\}$ be a finite cover of M by coordinate charts as before. Given an embedding f, we have for each $i \geq 1$ a positive a_i so that if $g \in C^1(M, N)$ and $||g - f||_{C^1(\overline{V_i})} < a_i$, then $g_{|\overline{V_i}}$ is an embedding. Since f is a homeomorphism from M to $f(M)$, we have $d_i = dist(f(\overline{W_i}), f(M \setminus V_i)) > 0$.

Choose the a_i so that $a_i < \frac{d_i}{3}$. We claim $C^1(f, a) \subset Emb^1(M, N)$ if $a = \min(a_i) > 0$. Clearly $C^1(f, a) \subset Imm^1(M, N)$. Show that any $g \in C^1(f, a)$ is injective.

Regarding the stability of regular values, we have the following.

Lemma. Let $K \subset M$ be compact, $\lambda : M \to R^s$ a C^1 map for which $0 \in R^s$ is a regular value, Then there exists $\delta = \delta(K) > 0$ so that if $\mu : M \to R^s$ is a C^1 map with $||\mu - \lambda||_{C^1(K)} < \delta$, then 0 is a regular value of $\mu_{|K}$.

20. Let S be a closed submanifold of N. Then if M is compact, the set of C^1 mappings $f : M \to N$ which are transversal to S is open in $C^1(M, N)$.

Follow the steps:

Let $\mathcal{C}$ be a covering of S by domains W of charts for N, $y : W \to R^n$, so that $y(W \cap S) \subset \pi^{-1}(0)$, where $\pi : R^n \to R^s$ projects on the last s coordinates (s is the codimension of S in N.)

Let $f \in C^1(M, N)$ be transversal to S.

Since S is closed in N, we may cover M by finitely many open sets (domains of charts): $M = \bigcup U_i$ finite, with charts $x_i : U_i \to R^m$ such that $x_i(U_i) = B(3)$ and, for a given i, either $f(U_i) \subset N \setminus S$, or $f(U_i) \subset W$, for some $W \in \mathcal{C}$. And $M = \bigcup \overline{V_i}, V_i = x_i^{-1}(B(2))$ is still a covering of M.

Given $i \geq 1$, there are two possibilities. The first is that $f(U_i) \cap S = \emptyset$. Since

$f(\overline{V_i})$ is compact and disjoint from the closed set S, we may choose $a_i > 0$ so that $||g - f||_{C^1(\overline{V_i})} < a_i$ implies $g(\overline{V_i}) \cap S = \emptyset$. Thus g is trivially transversal to S on $\overline{V_i}$.

The second possibility is that $f(U_i) \cap S \neq \emptyset$, so $f(U_i) \subset W$ for some $W \in \mathcal{C}$. Then since f is transversal to S, considering the chart $y : W \to R^n$ and the projection $\pi : R^n \to R^s$, we know that $0 \in R^s$ is a regular value of the map $\lambda = \pi \circ y \circ f : U_i \to R^s$. Then use the lemma to find $a > 0$ so that the basic neighborhood $C^1(f, a)$ of f consists only of maps $g : M \to N$ transversal to S.

Sard's Theorem. (i) Let $U \subset R^m$ open, $f : U \subset R^m$ smooth, $C \subset U$ the set of critical points of f. Then $f(U)$ has measure 0 in R^n.

Remark: the theorem is true for C^r maps if $r > \max\{0, m-n\}$. In particular, true for C^1 maps if $m \leq n$. And true for C^{m-1} real-valued functions of m variables, $m \geq 1$.

(ii) Let $f : X \to Y$ be a smooth map of manifolds. Then the set of critical values of f has measure 0 in Y.

21. (i) Define 'set of measure 0' in $\mathbb{R}^n$.

(ii) Show that if $A \subset R^n$ has measure zero and $f : R^n \to R^m$ is a Lipschitz map (where $m \geq n$), then $f(A)$ has measure zero in R^m.

(iii) Explain why the notion 'set of measure 0' makes sense on differentiable manifolds.

22. (i) Show that if $k < l$, $\mathbb{R}^k$ has measure zero in $\mathbb{R}^l$.

(ii) Suppose $Z \subset X$ is a submanifold with $dim(Z) < dim(X)$. Prove that Z is a set of measure 0 in X.

23. If $dim(X) < dim(Y)$, the image of any C^1 map $f : X \to$ is a set of measure zero in Y. (Prove this without using Sard's tehreom.)

24. (i) Prove that any smooth loop in S^n $(n > 1)$ is homotopic to the constant loop (with fixed basepoint.) *Hint: Sard's theorem.*

(ii) Prove S^n is simply connected if $n > 1$, using a covering by two simply-connected open sets, with connected intersection.

Remark. Given $f : S^1 \to S^n$ continuous, we may find $g : S^1 \to S^n$ of class C^1, so that $|g(x) - f(x)| < 2$, $\forall x \in S^1$. Thus f is homotopic to g. In fact:

25. Let X be a compact smooth manifold. Every continuous map $f : X \to S^n \subset R^{n+1}$ may be approximated by a smooth map, homotopic to f.

Hint. Assume $X \subset R^N$ (embedded), and use the Stone-Weierstrass theorem for each of the $n+1$ components of f to approximate f by a smooth map $g : X \to R^{n+1}$. Then normalize g, observing that $||g(x)|| > 1 - \epsilon$ if $||f(x) - g(x)|| < \epsilon$.

26. Let $f : M \to R^s$ be a C^1 map, $N \subset R^s$ a submanifold of codimension strictly greater than $dim(M)$, Then for almost every $v \in R^s$ the translated image $f(M) + v$ has empty intersection with N. (That is, the set of $v \in R^s$ for which the intersection is *not* empty has measure zero in R^s.)

27. If $dim(M) < p$, M compact, any C^1 map $f : M \to S^p$ is nullhomotopic.