PROBLEM LIST 9: FUNDAMENTAL GROUP AND COVERING SPACES (Version date: 7/8/2021)

1. (i) Covering maps are open maps.
(ii) Finite-sheeted covering maps are closed maps.
(iii) Give an example of a covering map that is not a closed map.

2. Construct two 4-sheeted covering maps $p_i : E_i \to S^1 \vee P^2$ (i=1,2) with E_1, E_2 connected, p_1 regular, p_2 not regular. Explain why they are covering maps and have the required properties.

3. Let $p : \tilde{X} \to X$ be a covering map, where X is a compact metric space. Prove that for some $\epsilon > 0$, every ball $B(x, \epsilon) \subset X$ is evenly covered.

4. Prove there are no covering maps from S^2 to $S^1 \times S^1$, or from $S^1 \times S^1$ to S^2.

5. Describe the three connected double coverings of $RP^2 \vee S^1$, and determine which ones (if any) are regular.

6. Let X be the union $S^2 \cup L$ of the standard sphere $S^2 \subset R^3$ and the closed vertical segment L from $e_3 = (0, 0, 1)$ to $-e_3$. Determine $\pi_1(X, e_3)$.

7. Let $X \subset R^3$ be the union of k distinct lines through the origin. Prove that $\pi_1(R^3 \setminus X)$ is a free group on $2k - 1$ generators.

8. Prove that every continuous map from RP^2 to T^2 is nullhomotopic.

9. Let $p : X \to Y$ be covering map, with Y connected. Show that if $p^{-1}(y_0)$ has k elements for some $y_0 \in B$, then $p^{-1}(y)$ has k elements for all $y \in B$.

10. Let $B = P^2 \vee P^2$.
(i) Find a connected three-sheeted covering space of B that is not regular, and use it to show the fundamental group of B is not abelian.
(ii) Find a connected, regular, infinite-sheeted cover of B, and use it to show the fundamental group of B is infinite.

11. Show that if $h : S^1 \to S^1$ is nullhomotopic, then h has a fixed point and h maps some point x to its antipode $-x$. (And similarly for all S^n)

12. Let $X = \bigcup_{n=1}^{\infty} X_n$, where each X_n is a simply-connected open subspace of X, and $X_n \subset X_{n+1}$ for each n, Show that X is simply connected.

13. Let B be a compact space, $p : E \to B$ a covering map. Show that if $p^{-1}(b)$ is finite for each $b \in B$, then E is compact.

14. Suppose a Hausdorff space X is the union of two simply-connected, path-connected open subsets U, V, with intersection $U \cap V$ non-empty and path-connected. Prove $\pi_1(X, x_0)$ is the trivial group, without using the Seifert-van Kampen theorem (unless you include a proof.) Here $x_0 \in U \cap V$ may be assumed.

15. Suppose there exists a deformation retraction from the space X to a point $x_0 \in X$. Show that for each open set U containing x_0, there exits a second open neighborhood $V \subset U$ of x_0, so that the inclusion homomorphism $i_* : \pi_1(V, x_0) \to \pi_1(U, x_0)$ is trivial. *Hint:* joint continuity of the deformation in (x, t).

16. Compute the fundamental group of the Klein bottle, by considering the Klein bottle as the quotient of the Euclidean plane under a suitable group action. (Prove that the action is properly discontinuous.)

Hint: Consider the group generated by the linear homeomorphism f, g of R^2: $f(x, y) = (x, y+1), g(x, y) = (x+1, 1-y)$. Show that $gf = f^{-1}g$ and that $f^m g^n(x, y) = (x+n, y+m)$ if n is even, and equals $(x+n, 1-y+m)$ if n is odd. Prove the action is properly discontinuous.

17. Let X be a path connected, locally path connected space whose fundamental group is finite. Prove that any map $f : X \to S^1$ is nullhomotopic.

18. Let $f : A \times B \to S^2$ be given by $f(x, y) = \frac{x-y}{||x-y||}$. Here:

$$A = \{(x, y, z) \in R^3; x^2+y^2 = 1, z = 0\} \quad B = \{(x, y, z) \in R^3; (y-3)^2+z^2 = 1, x = 0\}.$$

Is the map f homotopic to a constant map?

19. X is path-connected, $p : \tilde{X} \to X$ is a 3-fold covering map, the fundamental group of X is isomorphic to the cyclic group of order 2. Show that $\tilde{X}$ is not connected and some component of $\tilde{X}$ is homeomorphic to X.

20. Let B denote the closed unit disk in R^2, with boundary S^1. Let $f : B \to R^2$ be a continuous map, such that $f(S^1) \subset B$. Show that f has a fixed point in B. *Hint:* Consider the map $\phi(x) = \frac{x-f(x)}{||x-f(x)||}$ from S^1 to S^1. What is its degree?

21. Let $f : S^n \to S^n$ be a continuous map such that $||f(x) - x|| < 1$ for all $\in S^n$ (for the usual norm in R^{n+1}.) Must f be onto?

22. Let $q : X \to Y$ and $r : Y \to Z$ be covering maps. Assume all preimages $r^{-1}(z)$ are finite, for all $z \in Z$. Show that $p = r \circ q$ is a covering map.

23. (i) Define 'homotopy equivalent spaces' and 'deformation retraction', and show that if X deformation retracts to a subspace $A \subset X$, then X and A are homotopy equivalent.

(ii) Show that the figure-eight space X and the theta-space Y are homotopy equivalent. Are they homeomorphic?

$$X = \{(x, y) \in R^2; x^2 + (y-1)^2 = 1\} \cup \{(x, y) \in R^2; x^2 + (y+1)^2 = 1\}.$$

$$Y = \{(x, y) \in R^2 : x^2 + y^2 = 1\} \cup \{(x, 0) \in R^2; -1 \leq x \leq 1\}.$$

Hint: prove that both spaces are homotopy equivalent to R^2 minus two points.

24. Let $A \subset R^n$ be a path-connected subspace, Y a path-connected topological space, $h : A \to Y$ continuous. Show that if h extends to a continuous map $\hat{h} : R^n \to Y$, then the induced map $h_* : \pi_1(A) \to \pi_1(Y)$ is trivial.

25. (i) Let $X = [0, 1] \times [-1, 1]$. Consider the equivalence relation on X that identifies $(0, t) \sim (1, t)$ for each $t \in [-1, 1]$, with every other point equivalent only to itself. Show that the quotient space $Y = X/\sim$ is homeomorphic to $S^1 \times [-1, 1]$.

(ii) In a similar way, we may define the finite Möbius strip M as the space obtained from the rectangle $[0, 1] \times [-1, 1]$ by identifying the vertical boundaries while reversing orientation: $(0, t) \sim (1, -t)$, for $t \in [-1, 1]$. Show that the finite cylinder $S^1 \times [-1, 1]$ is a double cover of the Möbius strip.

Hint. Find a properly discontinuous action of $\mathbb{Z}_2$ on the cylinder with M as the quotient space (prove it is properly discontinuous.) To identify the quotient space with M, find an open rectangular subset of the cylinder which maps homeomorphically under the quotient projection, with vertical sides identified as described.

26. Let X be the quotient space of the closed 2-dimensional unit disk by the equivalence relation on the boundary:

$$z \sim ze^{2\pi i/3}, \quad |z| = 1.$$

(i) Compute the fundamental group of X; (see [Munkres], p. 444.)

(ii) Prove that every continuous map from X to the projective plane P^2 is nullhomotopic.

27. Let X be a path-connected space whose fundamental group is cyclic of order n (where n is prime), and let $p : \tilde{X} \to X$ be an $(n + 1)$-sheeted covering map. Show that $\tilde{X}$ is not path-connected, and that some path component of $\tilde{X}$ is homeomorphic to X.

28. Let C be a compact subset of the real line $\mathbb{R}$, and consider $f : C \to S^1$ continuous. Prove there exists $g : C \to \mathbb{R}$ so that $f(c) = e^{ig(c)}$, for all $c \in C$.

29. Let X be the figure eight, $\tilde{X}$ the subset of the upper half-plane made up of the horizontal axis, together with circles of radius $1/3$ tangent to it at the points of integral abscissa. Define a covering map from $\tilde{X}$ to X. Is this covering regular? With x_0 the origin in R^2, describe the conjugacy class of $\pi_1(X, x_0)$ defined by this covering, and determine $G = Aut(\tilde{X}|X)$.

30. Let X be an arbitrary topological space. Given the cover $p : R \to S^1, p(t) = e^{it}$, prove that a continuous map $f : X \to S^1$ has a lift relative to p if, and only if, f is homotopic to a constant.

Remark and hint: Note you can't use the lifting theorem, since that requires the domain to be connected and locally path connected. Instead, use the fact homotopies always lift over a covering map.

31. Let M be a compact orientable surface of genus $g > 1$. Prove there exists $f : M \to S^1$ continuous, not homotopic to a constant. *Hint:* show that M retracts to a wedge of g circles.

32. Suppose $f : S^1 \to R$ is continuous. Show there exists $z_0 \in S^1$ so that $f(-z_0) = f(z_0)$.

33. Let $p : E \to B$ be a map of topological spaces. Suppose U and V are open subsets of B evenly covered by p, with $U \cap V$ non-empty and connected. Show that $U \cup V$ is evenly covered by p.

34. (i) Let $p(z) = 2z^3 - 9z^2 - 12z + 1$. Find finite sets $F_1, F_2 \subset \mathbb{C}$, so that $p : \mathbb{C} \setminus F_1 \to \mathbb{C} \setminus F_2$ is a local diffeomorphism; show this implies it is a 3-sheeted covering.

(ii) Let $U \subset \mathbb{C}$ be the set of complex numbers $w = t + xi$ such that $t > 0$, X the set of real numbers $t \leq 0$, $V = \mathbb{C} \setminus X$. Show that $f : U \to V$, $f(w) = w^2$, is a local diffeomorphism, surjective and proper, hence a global diffeomorphism from U to V.

35. (i) Let $p : X \to Y$ be a surjective local homeomorphism. If X is compact, p is a covering map with finite fibers.

(ii) Give an example of a surjective local homeomorphism that is not a covering map.

(iii) If X, Y are compactly generated, a surjective local homeomorphism $p : X \to Y$ is a covering map with finite fibers if it is a proper map.

36. Let X, Y be metric spaces. A continuous surjective map $f : X \to Y$ is proper iff for any sequence (x_n) in X eventually leaving any compact subset of X, the sequence $f(x_n)$ in Y has the same property.

Theorem: covering homomorphisms are covering maps. Let $p_1 : \tilde{X}_1 \to X, p_2 : \tilde{X}_2 \to X$ be covering maps, with $\tilde{X}_i$ connected and locally path-connected. Assume $f : \tilde{X}_{\to}\tilde{X}_2$ is a covering homomorphism: $p_2 \circ f = p_1$. Then f is a covering map. If the induced homomorphism f_* on π_1 is surjective (and hence an isomorphism), then f is a homeomorphism.

37. Let X be a connected and locally path connected space, G a group of homeomorphisms acting properly discontinuously on X. Let $f : X \to X$ be a continuous map such that for all $x \in X$ one may find $g \in G$ so that $f(x) = gx$. Prove that f is a homeomorphism.

38. Let $\sim$ be an equivalence relation on the Hausdorff space X, and endow the quotient space $Y = X/\sim$ with a topology making the projection $\pi : X \to Y$ a continuous open map. Prove that Y is Hausdorff if and only if the graph $\Gamma = \{(x, y) \in X \times X; x \sim y\}$ is a closed subset of $X \times X$.