Newton's derivation of Kepler's laws (outline)

1. Brief history. The first known proposal for a heliocentric solar system is due to Aristarchus of Samos (ancient Greece, c. 270 BC). Following a long period of ignorance (appropriately known as the Dark Ages), the proposal was resurrected and detailed by the Polish 'amateur astronomer' Nicolaus Copernicus, apparently motivated more by reflection and reading of the ancients than observation. His masterwork *De revolutionibus orbium celestium* appeared in 1540, and Copernicus just barely lived to see the first printed copies. The proposal included circular orbits, but to reproduce the observed data these were complicated by 'epicycles' (now we know why.) Extensive observations were carried out by the Danish astronomer Tycho Brahe, over a period of 20 years, in the observatory-castle *Uraniborg*, on the island of Hven, generously financed by the Danish king (without a telescope-Brahe was a colorful personality, look it up on the web). In spite of that, Brahe's astronomical treatise (published in 1588) made negative theoretical progress relative to Copernicus. Upon the king's death, Brahe lost his funding and observatory, and moved to Prague. Luckily, in 1600 he hired the German Johannes Kepler as an assistant. Analyzing Brahe's Uraniborg data, Kepler came up with a completely accurate heliocentric system, embodied in his three 'laws' (the first two published in 1609, the third in 1619). Kepler already had the benefit of a telescope, as did another major early proponent of the heliocentric system, Galileo Galilei (the two were in correspondence). The latter made extensive astronomical observations, and his 'Dialogue concerning the two world systems'(1632) promoting the Copernican system made the Catholic Church unhappy (it can't have helped that the character defending the geocentric system was named 'Simplicio'). As is well known, Galileo was forced to recant and placed under house arrest (after all, Giordano Bruno had burned at the stake as recently as 1601 for similar 'heresies'). Anyway, we know how things turned out eventually (the Pope issued an official posthumous apology in 1993.) Gradually the Copernican system gained wide acceptance. The next major development is the publication of Isaac Newton's *Principia* (Mathematical Principles of Natural Philosophy, 1687), the foundational work of modern science. Newton used classical geometry and the emerging techniques of differential and integral calculus to give mathematical derivations from general fundamental principles of essentially all the quantitative physical laws known at the time. One of the main achievements was the derivation of Kepler's laws from an inverse-square law for gravity, the topic of this handout. In fact, this was the very result that forced Newton to write the *Principia*, which contained results he discovered over the previous two decades: he was prompted by news that Robert Hooke claimed he could derive the equation for the orbits of bodies attracted to another by an inverse-square law, a result Newton had obtained (but not published) more than a decade earlier. One result of the ensuing priority dispute was that Newton did not set foot in the Royal Society of London (of which Hooke at one point was president) until after Hooke's death.

For an entertaining, instructive account of the history of Cosmology (including details on the above and much more) read Simon Singh's book 'Big Bang' -paperback 2006.

In sections 2-8 we present Newton's derivation of Kepler's laws from the inverse-square law for gravity, which only uses basic calculus. (*Nothing* is original in the presentation that follows- it is all in the public domain, what with being 320 years old.) Of course, the notation is 'modern', and would not have been recognized by Newton.

2. Motion in the plane in polar coordinates. At each point in the plane (except the origin) we have an orthonormal frame given by the unit radial and 'unit θ' vectors:

$$\vec{u_r} = (\cos\theta, \sin\theta), \quad \vec{u_\theta} = (-\sin\theta, \cos\theta).$$

The position and velocity vectors have decompositions:

$$\vec{r}(t) = r\vec{u_r}, \quad \vec{r'}(t) = r'\vec{u_r} + r\theta'\vec{u_\theta}.$$

We use primes to denote derivatives with respect to t (time), in particular:

$$\vec{u}_r' = \theta'\vec{u}_\theta, \quad \vec{u_\theta'} = -\theta'\vec{u_r}.$$

Differentiating $\vec{r'}$, we obtain the decomposition of the acceleration vector:

$$\vec{r''} = (r'' - r(\theta')^2)\vec{u_r} + (2r'\theta' + r\theta'')\vec{u_\theta}.$$

3.The inverse-square law. Newton proposed that any two massive bodies attract each other gravitationally with a force directed along the line between their centers, of magnitude directly proportional to the product of their masses and inversely proportional to the square of their distances. If the mass M of the large body ('Sun') is much greater than that (m) of the small one ('planet'), it is an excellent approximation to assume that the Sun stays put (at the origin of the coordinate system), and gravity causes the planet to follow a path $\vec{r}(t)$. Since the force at position $\vec{r}$, $F(\vec{r})$, pulls the planet towards the origin, we have a (-) sign when expressing this 'law' analytically:

$$F(\vec{r}) = -\frac{GMm}{r^2}\vec{u_r}.$$

Three remarks: (i) in principle $\vec{u_r}$ denotes a unit radial vector *in space.*; (ii) The constant of proportionality G and solar mass M cannot be separately determined from Earth-based *astronomical* observations alone- the gravitational effect of the sun on the planets depends only on their product $M_G = GM$; (iii) Newton assumed that the parameter m in this law, measuring the 'quantity of matter' in the planet ('gravitational mass'), is the same as the constant of proportionality m ('inertial mass') in his *second law of motion* :

$$\vec{F} = m \cdot \vec{r}''.$$

That this assumption is not trivial was pointed out by Ernst Mach in the late 19th. century, eventually leading to one of the key ideas in Albert Einstein's geometric theory of gravitation (general relativity, 1916), the 'equivalence principle'.

4.Kepler's second law and angular momentum. Since the force $\vec{F}(\vec{r})$ has no component along $\vec{u}_\theta$, the $\vec{u}_\theta$ component of the acceleration must vanish for all t:

$$2r'\theta' + r\theta'' = 0.$$

Multiplying this by r, we get an exact derivative:

$$(r^2\theta')' = 2rr'\theta' + r^2\theta'' \equiv 0.$$

This can be expressed as a *conservation law.* The quantity:

$$L := r^2\theta'$$

is constant throughout the motion. This has a geometric interpretation: recall that the area enclosed by a curve given in polar coordinates by $r = r(\theta)$, between two given θ values, is:

$$A(\theta_1, \theta_2) = \int_{\theta_1}^{\theta_2} \frac{1}{2} r^2(\theta) d\theta.$$

This implies that the instantaneous rate (with respect to time) at which the 'area swept by the position vector' changes is:

$$\frac{dA}{dt} = \frac{1}{2} r^2(t)\theta' = \frac{1}{2} L \equiv const.$$

Thus we have *Kepler's second law:*

'The line from the Sun to a planet sweeps out area at a constant rate, as the planet moves on its orbit.'

This has an observational consequence (which is how the law was found): planets do not move at a constant speed (as Copernicus thought), but faster when closer to the sun, slower when farther out. To state the physical interpretation of the law, recall that for planar motion the angular momentum about the origin is defined by mrv_θ, where v_θ is the $\vec{u}_\theta$ component of the velocity, which as seen above equals $r\theta'$. Thus Kepler's second law is equivalent to conservation of angular momentum, and is true for any radial force (not necessarily following an inverse-square law).

Remark. (For readers who know something about vector products.) For motion in space, the angular momentum with respect to the origin is the vector:

$$\vec{L} = m\vec{r} \times \vec{r}'.$$

Its derivative with respect to time is (using the product rule and a basic property of vector products:)

$$\vec{L}' = m\vec{r} \times \vec{r}'' = r\vec{u_r} \times \vec{F}(\vec{r}),$$

and this vanishes if the force is radial ($\vec{F}(\vec{r}) = f(r)\vec{u_r}$.) In particular, we see that the normal to the instantaneous plane of motion at t (which has direction given by $\vec{r} \times \vec{r}'$) is constant in time. So motion under a central force in space takes place in a fixed plane (the plane spanned by $\vec{r}$ and $\vec{r}'$ at time 0.)

5. The orbit equation and Kepler's first law. Equating the gravitational force with the mass times the radial component of acceleration, and substituting $(\theta')^2 = L^2/r^4$, we find the *'orbit equation'*:

$$r'' - \frac{L^2}{r^3} = -\frac{M_G}{r^2}.$$

This is a 'second-order nonlinear differential equation' for the distance $r(t)$ as a function of time, and hard to solve. Amazingly, if we are just interested in a polar equation for the *orbit*, that is, in r as a function of θ, there is a trick to change it into a much simpler differential equation. Consider the reciprocal of r as a function of θ:

$$u(\theta) = \frac{1}{r(t(\theta))},$$

where we used the inverse function $t(\theta)$ of $\theta(t)$ (well-defined since $\theta'(t) > 0$ always; planets don't backtrack along their orbits). Using what you know about the chain rule and derivatives of inverse functions, you can derive (*exercise)*:

$$\frac{du}{d\theta} = -\frac{1}{L}r'(t), \quad \frac{d^2u}{d\theta^2} = -\frac{1}{L^2}r^2r''.$$

Substituting in the orbit equation, we obtain:

$$\frac{d^2u}{d\theta^2} + u = \frac{M_G}{L^2}.$$

Anyone who has taken a first course in differential equations can write down the general solution to this one immediately. Even if you haven't, it is easy enough to check that the function:

$$u(\theta) = A\cos\theta + \frac{M_G}{L^2}$$

is a solution, for any value of A. Taking the reciprocal of u, we find the equation for the orbit:

$$r(\theta) = \frac{L^2/M_G}{1 + \frac{AL^2}{M_G}\cos\theta}.$$

Now, if $A \neq 0$ and $AL^2/M_G < 1$, this is the polar equation of an ellipse with eccentricity AL^2/M_G and 'latus rectum' (length of the chord through the focus perpendicular to the axis) equal to L^2/M_G. This shows that an inverse-square law for gravity implies *Kepler's first law:*

'The orbits of the planets are ellipses, with the Sun at one focus'

Note that, while the latus rectum is given directly in terms of the physical parameters L and M_G, it is less obvious how to relate the geometric parameter A (the ratio of eccentricity to latus rectum, or $A = e/(ed) = 1/d$, where d is the distance from a focus to its directrix) to physical properties of the motion. This is done below (section 9).

Remarks: (i) In the *Principia*, Newton also proves the converse: if the orbits are ellipses, the magnitude of the force must follow an inverse-square law. (ii) The orbits of the planets are practically circular, with very small eccentricities: 0.2, 0.01, 0.02, 0.1, 0.05, 0.06, 0.05, 0.01 (from Mercury to Neptune). Still, the deviation from circularity apparently was large enough to be detected by Kepler from Brahe's data.

6. Kepler's third law. The first two laws appeared in Kepler's *Astronomia Nova* (1609), but the third had to wait another 10 years (*Harmonices mundi*, 1619). It states:

'The squares of the periods of the planets about the Sun are proportional to the cube of their mean distances to the Sun.'

This follows from two facts: first, as seen above, the area swept by the position vector increases at the constant rate (per unit time) equal to $L/2$. So the total area enclosed is $LT/2$, where T is the period. Since the area enclosed by an ellipse is πab, we have:

$$\frac{L^2}{4}T^2 = \pi^2 a^2 b^2 = \pi^2 a^4 (1 - e^2).$$

Now compare two expressions for the distance of closest approach to the Sun, r_{min}:

$$a(1 - e) = r_{min} = \frac{L^2/M_G}{1 + e}$$

to conclude: $a(1 - e^2) = L^2/M_G$. Substituting into the preceding relation, we have:

$$T^2 = \frac{4\pi^2}{M_G} a^3,$$

which confirms Kepler's law and gives the constant of proportionality.

The significance of Kepler's third law is that it amounts to a 'scale map' of the solar system. The point is that the periods are directly observable, so all the distances will be known once a single distance can somehow be measured to fix the scale. For example, the third law implies that the period measured in earth-years (t) and the mean distances to the Sun measured in AU (δ) obey the simple relation:

$$t = \delta^{3/2}.$$

By definition, one AU (astronomical unit) is the mean distance from Earth to the Sun. There were good low-tech estimates of this distance in the 19th century, using the observed parallax of Venus or Mercury during their solar 'transits'. Its value is approximately 150,000,000 km. For the distances in AU we have the approximate mnemonic (*'Titius-Bode law'*, 1766):

$$\delta = 0.4 + 0.3 \cdot 2^n,$$

with $n = -\infty$ for Mercury, $n = 1$ for Venus, $n = 2$ for Earth, etc. ($n = 4$ is the asteroid belt), fairly accurate except for Neptune, where δ is about 30 AU, not 38.8 AU as predicted. (Few people believe nowadays there is any deep physical reason behind this 'law'.)

7. Example: Earth-Moon system. One can use Kepler's third law to estimate the distance from Earth to its moon with a 'back of the envelope calculation'. The acceleration of gravity $g \sim 10m/s^2$ on the Earth's surface is related to Earth's radius R by the expression (derived from Newton's laws of gravity and of motion):

$$g = \frac{M_G}{R^2},$$

where M_G now refers to the Earth's mass. The value of R was already measured to a fair degree of accuracy by the Greek geometer Eratosthenes (in c. 250 BC): $R \sim 6,400$km (divide the length of the equator, $40,000$ km, by 2π). This gives for M_G the approximate value $M_G \sim 4 \times 10^{14}$. From the expression for Kepler's 3rd. law given above (with the constant of proportionality), we have for the distance a from Earth to the moon:

$$a^3 = \frac{M_G}{4\pi^2}T^2,$$

so we only need T, for which we may use the conventional value 30 days, or about 2.6×10^6 seconds. This gives for a the value $409,000$ km, compared with the standard modern value of $384,000$ km. Not bad for 'back of the envelope' (note that the Ancient Greeks already had all the data, but did not have Kepler's law.)

8. Example: Earth-Sun system. Using the back of the same envelope, we can estimate M_G for the sun, just by recalling the value given above for the AU and the number of seconds in a year, which upon squaring gives $T^2 \sim 10^{15}$:

$$M_G(sun) = \frac{4\pi^2}{T^2} a^3_{earth-sun} = \frac{4\pi^2}{10^{15}}(150 \times 10^9) \sim 1.33 \times 10^{20}$$

(in SI units).

The value of G was first measured in 1798 by Henry Cavendish, via a delicate experiment using a torsion balance; the official value of G is 6.67×10^{-11} SI units. With this value and the known values of M_G, one can estimate the masses of the earth and the sun (for example, with the above value we find 2×10^{30} kg for the solar mass).

9. Conservation of energy. (*This part not presented in Calculus II.*)

The inverse-square law for gravity corresponds to a potential energy (per unit mass of the orbiting planet) given by $-M_G/r$. The kinetic energy per unit mass is $(speed)^2/2$, or in terms of the components of velocity in polar coordinates: $[(r')^2 + r^2(\theta')^2]/2$. Using the conserved angular momentum L to eliminate $(\theta')^2$, we obtain for the total energy per unit mass (kinetic plus potential) the expression:

$$E = \frac{(r')^2}{2} + \frac{L^2}{2r^2} - \frac{M_G}{r}.$$

Exercise: Use the orbit equation to show that this is a constant of motion (i.e., its time derivative vanishes identically).

This expression can be thought of as the total energy of a particle of unit mass moving in the half-line $\{r > o\}$ under the 'effective potential':

$$V(r) = \frac{L^2}{2r^2} - \frac{M_G}{r}.$$

The 'bound orbits' (ellipses) correspond to initial data for which $V < 0$. If r_0 and v_0 are the distance to the sun and speed at some point on the orbit, the total energy and angular momentum (per unit mass of orbiting planed) are given by:

$$E = \frac{v_0^2}{2} - \frac{M_G}{r_0} + \frac{L^2}{2r_0^2}, \quad L = r_0^2 v_{\theta 0};$$

once E and L are known, the least and greatest distances to the Sun along the orbit, r_{min} and r_{max}, are obtained from the solutions of the quadratic equation in $1/r$:

$$\frac{L^2}{2r^2} - \frac{M_G}{r} = E$$

(assuming M_G is also known.) And then the geometric parameters a and e for the orbit are easily found, since:

$$r_{min} = a(1 - e), \quad r_{max} = a(1 + e).$$

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