COMPLETENESS OF THE REAL NUMBERS

1. **Need for the real numbers.** By extending $\mathbb{Z}$ to $\mathbb{Q}$, we ensure that linear equations:

$$ax = b, \quad a \in \mathbb{Z}_+, b \in \mathbb{Z}$$

have solutions in $\mathbb{Q}$. However, quadratic equations (or higher-degree polynomial equations) with integer coefficients often fail to have rational solutions. For example,

$$x^2 - 2 = 0$$

does not have a solution $x \in \mathbb{Q}$, a fact known to mathematicians in Ancient Greece. (*Proof.* By contradiction, suppose we had a solution $x = p/q > 0$, with $p, q \in \mathbb{N}$, both nonzero and with no common factors. Then $p^2 = 2q^2$ shows p^2 is even, hence p is even, $p = 2k$ with $k \in \mathbb{N}^*$. But then $4k^2 = 2q^2$, so $q^2 = 2k^2$ and q^2 is even. Thus q is also even, contradicting the assumption that p, q have no common factors.

More generally, we have the following criterion for rational roots.

Theorem. Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ be a polynomial with integer coefficients: $a_i \in \mathbb{Z}, N \geq 1, a_n \neq 0, a_0 \neq 0$. Assume $r = p/q$ is a rational root, where $p \in \mathbb{Z}$ and $q \in \mathbb{Z}^*$ have no common factors. Then p divides a_0 and q divides a_n.

Proof. Writing down the fact that $p(r) = 0$ and clearing denominators, we find:

$$a_n p^n + a_{n-1}p^{n-1}q + \ldots + a_1 p q^{n-1} + a_0 q^n = 0.$$

Thus, on the one hand:

$$a_n p^n = -(a_{n-1}p^{n-1}q + \ldots + a_1 p q^{n-1} + a_0 q^n).$$

Thus shows q divides $a_n p^n$, and since p, q have no common factors q must divide a_n. On the other hand:

$$a_0 q^n = -(a_n p^n + a_{n-1}p^{n-1}q + \ldots + a_1 p q^{n-1}),$$

so p divides $a_0 q^n$, and therefore p divides a_0.

Problem 1. Show that $5^{1/3}$ is not rational (assuming such a number exists!)

Remark. It takes more work to show the numbers π (defined in geometry, or trigonometry) and e (defined in Calculus) are irrational (=not rational).

In fact much more is true, these numbers are *transcendental*: they are not roots of any polynomial with integer coefficients, of any degree.

Thus if we want odd-order roots of rational numbers (or even-order roots of positive rational numbers) to exist, we need to extend $\mathbb{Q}$ to a bigger ordered field containing it, and satisfying an additional property. Recall:

Definition. Let F be an ordered field, $A \subset F$ a non-empty subset bounded from above. A number $L \in F$ is the *supremum* of A if:

$$(i)(\forall a \in A)(a \leq L) \text{ (L is an upper bound for A).}$$

$$(ii)(\forall L' \in F)(L' \text{ is an upper bound for } A \to L \leq L').$$

2. The supremum property. It is far from intutitive, but the property that works is the following.

Definition. An ordered field $(F, +, \cdot, 0, 1, F_+)$ has the *supremum property* if any non-empty subset $A \subset F$ bounded from above has a supremum $L = \sup(A) \in F$.

From now on we denote by $\mathbb{R}$ (the real numbers) an *ordered field with the supremum property.* We haven't proved such a field *exists*, or that there is only one (in a certain sense.) But we'll proceed under the assumption that both existence and uniqueness hold. We denote by $\mathbb{Q}$ the rational subfield of $\mathbb{R}$.

We next show that the supremum property fails for the rational numbers, and show the existence of a positive solution to $x^2 - 2 = 0$ using the supremum property for $\mathbb{R}$.

Lemma. Let $a, b, c \in \mathbb{Q}_+$ with $a < b^2 < c$. Then there exist $x, y \in \mathbb{Q}_+$ such that $x < b < y$ and $a < x^2, y^2 < c$.

Proof. Let $y = b + \epsilon$ with $\epsilon = min\{\frac{1}{2}, \frac{c-b^2}{2b+1}\}$. Let $x = b - \delta$, with $0 < \delta < \frac{b^2-a}{2b}$.

Proposition. Consider the set $A = \{x \in \mathbb{Q}_+ | x^2 < 2\}$. A is non-empty and bounded above. Let $L = sup(A) \in \mathbb{R}$ (which exists from the supremum property). Then $L^2 = 2$.

Proof. (i) Assume $L^2 < 2$. By the lemma, we may find $y \in \mathbb{Q}_+$ with $y > L$ and $y^2 < 2$. Thus $y \in A$, so L can't be an upper bound for A. (ii) Assume $L^2 > 2$. From the lemma, we find $x < L$ so that $x^2 > 2$. Then if $a \in A$, $a^2 < 2 > x^2$, so $a < x$. Thus x is an upper bound for A, and it is smaller than L, contradicting $L = sup(A)$. Thus we must have $L^2 = 2$. (In particular, $L \notin \mathbb{Q}$).

Problem 2. Show that the supremum property implies the *infimum property*: a non-empty set of real numbers which is bounded below has an infimum. *Hint:* show that if $-A = \{-x | x \in A\}$, then A is bounded below iff $-A$ is bounded above, and $L = inf(A)$ iff $-L = sup(-A)$.

Problem 3. Let $R \in \mathbb{Q}_+$ be a positive rational number. Prove that there exists $x \in \mathbb{R}_+$ such that $x^2 = R$. *Hint:* Repeat the proof given for $R = 2$, using the lemma above.

3. The Archimedean property, density and approximations. We begin by proving the Archimedean property for $\mathbb{R}$, using the supremum property.

Theorem. Let $a, b \in \mathbb{R}_+$. Then $(\exists n \in \mathbb{N})(na > b)$.

Proof. If $a > b$, just let $n = 1$. So we may assume $a \leq b$. Consider the set:

$$A = \{k \in \mathbb{N} | ka \leq b\} \subset \mathbb{R}.$$

Since $1 \in A$, A is nonempty. Clearly $b/a \in \mathbb{R}_+$ is an upper bound for A. Let $k_0 = sup_{\mathbb{R}} A$. From the lemma that follows, $k_0 \in A$ is the largest element of A, so $k_0 + 1 \notin A$, and $(k_0 + 1)a > b$.

Lemma. Let $A \subset \mathbb{N}$ be non-empty, and bounded above. Then A has a largest element (hence $sup_{\mathbb{R}}(A) \in A$.)

Proof. Let $L = sup_{\mathbb{R}}(A)$. Since $L - 1$ is not an upper bound for A, there exists $a_0 \in A$ so that $L - 1 < a_0$, or $L < a_0 + 1$. Hence $a_0 + 1 \notin A$. We *claim* that $a_0 = max(A)$, the largest element of A.

Since $A \subset \mathbb{N}$, given $a \in A$ either $a \leq a_0$ or $a \geq a_0 + 1$. The latter is impossible, since it would imply $a > L$, an upper bound for A. Thus $a_0 \in A$ is an upper bound for A, and hence is the largest element of A.

In particular, if $b \in \mathbb{R}$ and $b < a_0$, b can't be an upper bound for A. Thus $a_0 = sup_{\mathbb{R}}(A)$, so $a_0 = L$.

It then follows (see handout 6) that $\mathbb{R}$ has the 'density property':

Definition. An ordered field F has the *density property* if given $a < b$ in F, one may find $c \in Q$ (the rational subfield of F) so that $a < c < b$.

We already proved (see handout 6) that the Archimedean property implies the density property. The converse is also true.

Problem 4. If an ordered field F satisfies the density property, it also satisfies the Archimedean property.

Hint. Let $x \in F$ be positive. By the density property, there exists $r \in Q_+$ so that $x < r < x+1$. Now use the fact that the rational subfield $Q \subset F$ satisfies the Archimedean property.

Sequences and approximations. A *sequence* $(x_n)_{n \geq 0}$ with values in a set X is a function $x : \mathbb{N} \to X$, where we denote $x(n)$ by x_n. When $X = F$ (an ordered field), and $L \in F$, we say $x_n \to L$ (or $\lim x_n = L$, x_n converges to L) if:

$$(\forall \epsilon > 0)(\exists N \in \mathbb{N})(\forall n \geq N) L - \epsilon < x_n < L + \epsilon.$$

Definition. An ordered field F has the *monotone approximation property* if given any $x \in F$ we may find an increasing sequence $(r_n)_{n \geq 0} \in Q$ (the rational subfield of F) so that $r_n \to x$. This means:

$$(\forall \epsilon > 0)(\exists N \in \mathbb{N})(\forall n \geq N) x - \epsilon < r_n \leq x.$$

Proposition. Assume the density property holds (for $\mathbb{Q} \subset \mathbb{R}$). Let $x \in \mathbb{R}$. Then we may find a sequence $(r_n)_{n \geq 1}$ of rational numbers such that $r_n \leq r_{n+1}$ and $\lim r_n = x$.

Proof. Using the density property, we find $r_1 \in \mathbb{Q}$ so that $x - 1 < r_1 < x$. Let $s_2 \in \mathbb{Q}$ be such that $x - \frac{1}{2} < s_2 < x$, and let $r_2 = \max\{s_2, r_1\}$. Then $r_2 \in \mathbb{Q}, r_1 \leq r_2$ and $x - \frac{1}{2} < r_2 < x$.

Inductively, assuming the rational numbers $r_1, \ldots, r_n$ have been chosen, we pick r_{n+1} as follows: choose $s_{n+1} \in \mathbb{Q}$ so that $x - \frac{1}{n+1} < s_{n+1} < x$ and let $r_{n+1} = \max\{s_{n+1}, r_n\}$. Then $r_n \leq r_{n+1}$ and, for each $n \geq 1$: $x - \frac{1}{n} < r_n < x$. Given $\epsilon > 0$, let $N \in \mathbb{N}$ be such that $\frac{1}{N} < \epsilon$. Such an N exists by the Archimedean property (see problem 5(ii), which , as seen above, is equivalent to the density property. Then $x - \epsilon < x - \frac{1}{N} < r_N < x$. So for all $n \geq N$:

$$x - \epsilon < r_N \leq r_n < x.$$

Since ϵ is arbitrary, this shows $\lim r_n = x$.

The converse is also true:

Proposition. Assume the monotone approximation property holds (for the subset $\mathbb{Q}$ of $\mathbb{R}$). Then given two real numbers $x < y$, we may find $r \in \mathbb{Q}, x < r < y$.

Proof. Assume first $y \notin \mathbb{Q}$. Let $r_n \in \mathbb{Q}$ be a monotone sequence (increasing, $r_n \leq r_{n+1}$) converging to y. Then $r_n < y$. Letting $\epsilon = (y-x)/2 > 0$, from the definition of limit we may find $N \in \mathbb{N}$ so that $|y - r_N| < \epsilon$. Thus:

$$x < \frac{x+y}{2} = y - \epsilon < r_N < y,$$

as desired (set $r = r_N$.) .

If $y \in \mathbb{Q}$ and $x \notin \mathbb{Q}$, use the existence of a *nonincreasing* sequence converging to x (so $x < r_{n+1} \leq r_n$) and a similar argument. If both $x, y \in \mathbb{Q}$, let $r = (x+y)/2$.

Remark. The arguments used in the proofs of the preceding propositions work in an arbitrary ordered field. We conclude that, in an arbitrary ordered field F, the Archimedean property (which says the integer sub-domain of F is unbounded in F), the density property and the monotone approximation property (which refer to the rational subfield) are all equivalent. The supremum property implies any of them.

Problem 5. In a general ordered field F, the following conditions are equivalent:

(i) Archimedean property: $(\forall a \in F_+)(\exists n \in Z_+^*) n > a$ (where Z is the integer sub-domain of F.)

(ii) $(\forall \epsilon \in F_+^*)(\exists n \in Z_+^*)(\frac{1}{n} < \epsilon)$.

(iii) If $x \in F$ is such that $(\forall n \in Z_+^*)(|x| \leq \frac{1}{n})$, then $x = 0$.

Problem 5B. Use the Archimedean property of $\mathbb{R}$ to prove that if $x \in \mathbb{R}, x \geq 0$ and $x < \frac{1}{n}$ for all $n \in \mathbb{N}^*$, then $x = 0$.

Thus the fact we can divide an interval in $\mathbb{R}$ into subintervals as small as desired is equivalent to the fact that $\mathbb{Z}$ is unbounded in $\mathbb{R}$.

Problem 5C. Let (x_n) be a sequence of real numbers such that $\lim x_n = L > 0$.

(i) Show that $(\exists N \in \mathbb{N})(\forall n \geq N) x_n \geq L/2$.

(ii) Use part (i) to prove that $\lim \frac{1}{x_n} = \frac{1}{L}$.

4. Monotone sequences and continued fractions.

Definition. An ordered field F has the *monotone convergence property* if any sequence (x_n) with values in F which is (i) monotone increasing ($x_n \leq x_{n+1}$) and (ii) bounded above necessarily has a limit.

Problem 8. Show this is equivalent to requiring that any monotone decreasing sequence ($x_{n+1} \leq x_n$) which is bounded below have a limit.

Theorem. In any ordered field F, the supremum property implies the monotone convergence property.

Proof. Given a monotone increasing sequence (x_n) which is bounded above, let $L = \sup(\{x_n; n \geq 0\})$. It is easy to show that $x_n \to L$.

The converse is true!

Theorem. Let F be an odered field with the monotone convergence property. Then if $A \subset F$ is non-empty and bounded above, there exists $L \in F$ so that $L = \sup(A)$.

Proof. (i) Let $a_1 \in A$ be arbitrary, $M_1 \in F$ an upper bound for A (so $a_1 \leq M_1$). If $(a_1 + M_1)/2$ is an upper bound for A, we set $M_2 = (a_1 + M_1)/2 \leq M_1, a_2 = a_1$. If it isn't, we may find $a_2 \in A$ so that $a_2 > (a_1 + M_1)/2 \geq a_1$, and set $M_2 = M_1$.

Inductively, given $a_n \in A$ and M_n an upper bound for A, we let $M_{n+1} = (a_n + M_n)/2 \leq M_n$ and $a_{n+1} = a_n$ if $(a_n + M_n)/2$ is an upper bound for A, and $M_{n+1} = M_n, a_{n+1} \in A, a_{n+1} > (a_n + M_n)/2 \geq a_n$ otherwise.

(ii) This way we build two sequences, $a_n \in A$, $a_n \leq a_{n+1}$, and M_n, $M_{n+1} \leq M_n$, where each M_n is an upper bound for A. From the hypothesis of the theorem, we have the existence of $\lim a_n = \alpha$ and $\lim M_n = M$. It is easy to see that $a_n \leq \alpha$ and $M \leq M_n$ for all n, and that $\alpha \leq M$.

(iii) M is an upper bound for A: if we had $a \in A$ with $a > M$, letting $\epsilon = a - M > 0$ we'd find $n \in \mathbb{N}$ so that $M_n < M + \epsilon = a$, contradicting the fact that M_n is an upper bound for A.

Also, no number $L < \alpha$ can be an upper bound for A: letting $\epsilon = \alpha - L$ we find $a_n \in A$ so that $a_n > \alpha - \epsilon = L$.

Thus if we show $\alpha = M$, it follows that this number is the least upper bound of A.

(iv) We prove $\alpha = M$ by contradiction: assume $\alpha < M$. Then $\alpha < \frac{\alpha+M}{2} < M$. Since $\lim \frac{a_n+M_n}{2} = \frac{\alpha+M}{2}$, letting $\epsilon = \frac{M-\alpha}{2} > 0$, we find $n \in \mathbb{N}$ so that:

$$\alpha = \frac{\alpha + M}{2} - \epsilon < \frac{a_n + M_n}{2} < \frac{\alpha + M}{2} + \epsilon = M.$$

There are two possibilities: if this $\frac{a_n+M_n}{2}$ is an upper bound for A, then (by construction of the sequence (M_n)) $M_{n+1} = \frac{a_n+M_n}{2}$, leading to $M_{n+1} < M$, a contradiction. If it isn't, then $a_{n+1} > \frac{a_n+M_n}{2} > \alpha$, again a contradiction.

This concludes the proof.

Remark. Note that the Archimedean property was *not* used in the proof. We conclude:

Corollary. In any ordered field F, the Supremum Property and the Monotone Convergence Property are equivalent; either of them implies the Archimedean Property.

Example: continued fraction approximations. Consider the sequence of positive rational numbers defined inductively as follows:

$$x_0 = 1, \quad x_{n+1} = 1 + \frac{1}{1 + x_n}.$$

Thus $x_1 = 3/2, x_2 = 7/5, \ldots$.

Problem 6. (i) Show that:

$$x_{n+2} - x_{n+1} = \frac{x_n - x_{n+1}}{(1 + x_{n+1})(1 + x_n)}.$$

Conclude that the sequence "oscillates": if $x_n > x_{n+1}$, then $x_{n+2} > x_{n+1}$.

(ii) Show that $|x_{n+2} - x_{n+1}| < |x_n - x_{n+1}|$, and that this implies:

$$x_1 < x_3 < x_5 < \ldots < x_6 < x_4 < x_2.$$

The subsequence of odd-order terms is increasing and the subsequence of even-order terms is decreasing.

(iii) Show that $x_n < 2, \forall n \geq 0$.

Problem 7. Show that, if the sequence x_n has a limit L, then $L^2 = 2$.

Thus we have an explicit approximation of $\sqrt{2}$ by rational numbers, *if* we can show the sequence converges.

By the Monotone Convergence Property for $\mathbb{R}$, we conclude $\lim x_{2n} = M$ and $\lim x_{2n+1} = L$ exist in $\mathbb{R}$, with $L \leq M$.

Problem 8. Show that $L^2 = 2$ and $M^2 = 2$. Thus $L = M$.

Problem 9. Here we find rational approximations to the positive root of $x^2 = R$, where $R > 1$ is a rational number. Consider the sequence $(x_n)_{n \geq 0}$ defined recursively by:

$$x_0 = 1, \quad x_{n+1} = 1 + \frac{R - 1}{1 + x_n}.$$

Show that $L = \lim x_n$ exists and satisfies $L^2 = R$.

Remark. The case $0 < R < 1$ may be reduced to this, since $x^2 = 1/R$ iff $x^{-2} = R$. Also the general quadratic equation:

$$y^2 + by = c, \text{ with } R = c + \frac{b^2}{4} \in \mathbb{Q}_+$$

may be reduced to this by the change of variable $x = y + \frac{b}{2}$.

5. Cauchy sequences. Let F be an ordered field. A sequence $(x_n)_{n\geq 0}$ of numbers in F is a *Cauchy sequence* if:

$$(\forall \epsilon > 0)(\exists N \in \mathbb{N})(m, n \geq N \to |x_n - x_m| < \epsilon).$$

Remark. Note that this definition makes sense in an arbitrary field with an absolute value $|\cdot|$, not necessarily ordered (and indeed in much greater generality.) This is the main reason to define 'completeness' using Cauchy sequences.

Definition. An ordered F is *complete* if any Cauchy sequence with values in F has a limit in F. (Alternatively, 'F has the Cauchy convergence property'.)

Theorem. Let F be an ordered field with the supremum property. Then F is complete.

Proof. We use the fact that the supremum property implies the Archimedean property and the monotone convergence property.

Let (x_n) be a Cauchy sequence. Letting $\epsilon = 1$ in the definition, we find N so that:

$$|x_n| \leq |x_N - x_n| + |x_N| < 1 + |x_N| \text{ for } n \geq N,$$

and thus:

$$|x_n| \leq \max\{|x_1|, \ldots, |x_{N-1}|\} + |x_N| + 1 \text{ for all } n \geq 1;$$

that is, Cauchy sequences are *bounded* (above and below).

Consider the auxiliary sequences formed from (x_n):

$$y_n = \inf\{x_m; m \geq n\} \quad z_n = \sup\{x_m; m \geq n\}.$$

(Here we used the supremum property.) Clearly (y_n) is monotone increasing ($y_n \leq y_{n+1}$), (z_n) is monotone decreasing ($z_{n+1} \leq z_n$) and $y_n \leq x_n \leq z_n$ for each $n \geq 1$. Thus the following limits exist:

$$\lim y_n = L = \sup\{y_n; n \geq 1\}, \lim z_n = M = \inf\{z_n; n \geq 1\}; L \leq M.$$

From the definition of Cauchy sequence, given $\epsilon > 0$ we may find $N \in \mathbb{N}$ so that, for $m \geq N$:

$$x_m > x_N - \epsilon \text{ and } x_m < x_N + \epsilon.$$

By definition of y_N and z_N, this implies:

$$y_N > x_N - \epsilon \text{ and } z_N < x_N + \epsilon.$$

Thus, since $M \leq z_N$ and $L \geq y_N$ for each $N \geq 1$, we have:

$$0 \leq M - L \leq z_N - y_N < x_N + \epsilon - (x_N - \epsilon) = 2\epsilon.$$

Since $\epsilon > 0$ is arbitrary, it follows (using the Archimedean property, see problem 5) that $L = M$. And then, passing to the limit in the inequality $y_n \leq x_n \leq z_n$ ("squeezing principle", Problem 9A below), we have $\lim x_n = L$.

Problem 9A. ("Squeezing principle".) Let $(x_n), (y_n), (z_n)$ be three sequences of real numbers satisfying, for each $n \geq 1$: $x_n \leq y_n \leq z_n$. Show that if $\lim x_n = \lim z_n = L$, it follows that $\lim y_n$ exists and equals L.

The converse is also true, *in the presence of the Archimedean property:*

Theorem. Let F be a complete Archimedean ordered field (Cauchy sequences converge). Then F has the Monotone Convergence Property: if (x_n) is a sequence in F, monotone increasing ($x_n \leq x_{n+1}$) and bounded above, then $\lim x_n$ exists.

This follows immediately from the next result:

Proposition. If F is an ordered field with the Archimedean property, any monotone increasing sequence which is bounded above is a Cauchy sequence.

Proof. Let M be an upper bound for $\{x_n; n \geq 1\}$: $x_n < M$ for all $n \geq 1$. If (x_n) is not Cauchy, we may find $\epsilon_0 > 0$ so that for every N there exist $m, n \geq N$ with $|x_m - x_n| \geq \epsilon_0$. Since the sequence is monotone increasing, this means we may find sequences m_j, n_j of natural numbers (both unbounded) so that:

$$m_1 < n_1 < m_2 < n_2 < m_3 < n_3 < \ldots, \quad x_{n_j} - x_{m_j} \geq \epsilon_0 \text{ for each } j \geq 1.$$

Thus the sum of the lengths of the disjoint open intervals $(x_{m_1}, x_{n_1}), \ldots, (x_{m_j}, x_{n_j})$ is equal to at least $j\epsilon_0$, and this implies: $x_{n_j} - x_{m_1} > j\epsilon_0$, for each $j \geq 1$. Using the Archimedean property, choose $j \in \mathbb{N}$ so that $j > (M - x_{m_1})/\epsilon_0$. Then $x_{n_j} - x_{m_1} > M - x_{m_1}$, or $x_{n_j} > M$, contradicting the fact that M is an upper bound for the sequence.

Problem 10. Show that any convergent sequence of real numbers is a Cauchy sequence. (*Hint:* by the triangle inequality, $|x_m - x_n| \leq |x_m - L| + |x_n - L|$.)

6. Decimal expansions of real numbers.

If $x \in \mathbb{R}$, $x > 0$, a decimal expansion of x is an infinite sequence:

$$d_0, d_1, d_2, \dots, \text{ written } x = d_0.d_1d_2d_3, \dots,$$

where $d_0 \in \mathbb{N}$, d_i for $i \geq 1$ is a natural number $0 \leq d_i < 10$, and the sequence y_n of rational numbers:

$$y_n = d_0 + \frac{d_1}{10} + \frac{d_2}{10^2} + \dots + \frac{d_n}{10^n}, n \geq 1,$$

satisfies: $\lim y_n = x$.

Unfortunately, if $x > 0$ is a rational number it admits *two* decimal expansions. For example,

$$0.123400000... = 0.12339999999...$$

It is important to understand that the $=$ sign has its usual meaning here, that is: this equality is *exact*, not *approximate!* To see this, let L be the real number defined by the symbol on the left, and R the real number defined by the symbol on the right. By definition of these symbols:

$$L = \frac{1}{10} + \frac{2}{10^2} + \frac{3}{10^3} + \frac{4}{10^4}, R = \frac{1}{10} + \frac{2}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + 9\sum_{j=5}^{\infty} \frac{1}{10^j},$$

thus:

$$10^4 L = 1234, 10^4 R = 1233 + 9\sum_{j=1}^{\infty} 10^{-j},$$

while:

$$9\sum_{j=1}^{\infty} \frac{1}{10^j} = 9 \lim_n \sum_{j=1}^{n} \frac{1}{10^j} = \frac{9}{10} \lim_n \frac{1 - 10^{-n}}{1 - 10^{-1}} = 1.$$

This clearly implies $L = R$. In fact, the last line above is just the proof that:

$$0.999999... = 1 \quad (EXACT!)$$

In the following, we choose to work with the symbol that *does not* 'end in 9s'. To do this, for any real number $x > 0$, denote $[x] = \max\{n \in \mathbb{N}; n \leq x\}$. Set $d_0 = [x]$. Then:

$$10x = 10d_0 + x_1, \text{ where } x_1 = 10(x - d_0) \in [0, 10) \subset \mathbb{R}.$$

Let $d_1 = [x_1]$, a natural number $0 \leq d_1 < 10$. Then:

$$10x_1 = 10d_1 + x_2, \text{ where } x_2 = 10(x_1 - d_1) \in [0, 10) \subset \mathbb{R},$$

so $d_2 = [x_2]$ is a natural number $0 \leq d_2 < 10$ and:

$$x = d_0 + \frac{x_1}{10} = d_0 + \frac{d_1}{10} + \frac{x_2}{10^2}$$

Proceeding in this fashion, assume $d_1, d_2, \ldots, d_n$ have been found (natural numbers $0 \leq d_i < 10$), with:

$$x = d_0 + \frac{d_1}{10} + \frac{d_2}{10^2} + \ldots + \frac{d_n}{10^n} + \frac{x_{n+1}}{10^{n+1}}, \text{ with } x_{n+1} \in [0, 10) \subset \mathbb{R}.$$

Then write:

$$d_{n+1} = [x_{n+1}], \quad 10x_{n+1} = 10d_{n+1} + x_{n+2}, \text{ with } x_{n+2} \in [0, 10) \subset \mathbb{R},$$

so:

$$x = d_0 + \frac{d_1}{10} + \frac{d_2}{10^2} + \ldots + \frac{d_n}{10^n} + \frac{d_{n+1}}{10^{n+1}} + \frac{x_{n+2}}{10^{n+2}}, \text{ with } x_{n+2} \in [0, 10) \subset \mathbb{R}.$$

Thus, letting $y_n = d_0 + \frac{d_1}{10} + \frac{d_2}{10^2} + \ldots + \frac{d_n}{10^n}$, we have, for each $n \geq 1$:

$$0 \leq x - y_n = \frac{x_{n+1}}{10^{n+1}} < \frac{1}{10^n},$$

so $\lim y_n = x$.

Problem 11. Implement this procedure for the example given at the beginning of this section and verify we obtain the representation that does not 'end in 9s'.

It is not hard to see that this choice of decimal expansion sets up a bijective correspondence between the interval $[0, 1) \subset \mathbb{R}$ and the set of all sequences (d_n) of natural numbers between 0 and 9 (inclusive) with the property: $(\forall N \in \mathbb{N})(\exists n > N) d_n \neq 9$.

Problem 12. Show that the set of rational numbers in $[0, 1)$ is in bijective crrespondence with the set of all eventually periodic sequences with values in $\{0, \ldots, 9\}$, which are not eventually equal to 9.

Choosing any 'basis' (a positive natural number) b instead of 10 yields a similar expansion, with $0 \leq d_i < b$. In particular, for $b = 2$ we obtain sequences of 0s and 1s, which are not eventually 1. That is:

Proposition: There exists a bijective correspondence between the real numbers $x \in [0,1)$ and the set:

$$\{d : \mathbb{N} \to \{0,1\}; (\forall N \in \mathbb{N})(\exists n > N) d_n \neq 1\}.$$

The set of binary sequences $d : \mathbb{N} \to \{0,1\}$ which are eventually periodic, but not eventually 1, is in bijective correspondence with $\mathbb{Q} \cap [0,1)$.