MAPPING PROPERTIES OF CONTINUOUS FUNCTIONS.

Let $f : I \to \mathbb{R}$ be continuous, where the domain I is assumed to be an open interval (possibly unbounded, or the whole real line). We consider whether various topological properties of subsets of $\mathbb{R}$ are preserved when we take image or preimage under f.

1. Openness. If $B \subset \mathbb{R}$ is open, the preimage $f^{-1}(B)$ is an open subset of I. This is equivalent to the statement that f is continuous. (*Prove this: 4.4.11.*)

In general, if $A \subset I$ is open, $f(A)$ is not open (*example?*) But this is true if f is strictly monotone. *Prove this.*

2. Connectedeness. If $A \subset I$ is an interval, then $f(A)$ is an interval (Intermediate Value Theorem). On the other hand, if $B \subset \mathbb{R}$ is an interval, $f^{-1}(B)$ is not necessarily an interval (*example?*) But this is true if f is strictly monotone. *Prove this!*

Recall that $A \subset R$ is an *interval* if:

$$(\forall x, y, z)(x \in A, y \in A \text{ and } x \leq z \leq y) \Rightarrow z \in A.$$

(Note this allows one-point sets, regarded as "degenerate intervals".)

Remark. A direct proof of this was given in class, but here is a lazy (yet "deeper") proof: if f is strictly monotone, it is injective, and hence defines a bijection from I to the image $J = f(I)$, which is also an open interval. Hence the inverse function $g = f^{-1} : J \to I$ is defined and (as shown in class) also continuous. But if $B \subset \mathbb{R}$ is an interval, so is $B \cap J$, and then $f^{-1}(B) = f^{-1}(B \cap J) = g(B \cap J)$ is also an interval (applying the I.V.T. to g.)

3. Compactness. If $K \subset I$ is compact, $f(K)$ is compact. (Thm 4.4.2, Extreme Value Theorem.) On the other hand, the preimage $f^{-1}(L)$ of a compact set $L \subset \mathbb{R}$ is not necessarily compact, even if f is strictly monotone. (*Example?*)

This is one of these cases where it would be really nice if this were true, so it gives rise to a definition, identifying an important class of continuous functions.

Definition. A continuous function $f : \mathbb{R} \to \mathbb{R}$ is *proper* if, for all compact sets $K \subset \mathbb{R}$, the preimage $f^{-1}(K)$ is also compact.

Problem 1. f is proper if, and only if, $|f(x)| \to \infty$ as $|x| \to \infty$. (Recall this means: $(\forall M > 0)(\exists R > 0)(\forall x)(|x| > R \Rightarrow |f(x)| > M)$.

Solution. Assume f is proper. If there was an $M > 0$ so that for all $R > 0$ we could find x so that $|x| > R$ and $|f(x)| \leq M$, it would be possible to find a sequence $x_n \to \infty$ (or $x_n \to -\infty$) of points in the preimage $f^{-1}([-M, M])$ of the compact set $K = [-M, M]$. Since this preimage is compact, such a sequence would have a convergent (in particular, bounded) subsequence, contradicting $|x_n| \to \infty$.

Conversely, if the condition is satisfied let K be a compact set (in particular bounded, hence contained in $[-M, M]$ for some $M > 0$). The condition then implies there exists an $R > 0$ so that $f^{-1}(K) \subset [-R, R]$. So any sequence (x_n) in $f^{-1}(K)$ is bounded, hence has a convergent subsequence: $\lim x_{n_j} = x$. Then $f(x) = \lim f(x_{n_j}) \in K$, since K is closed, so $x \in f^{-1}(K)$. By definition, this means $f^{-1}(K)$ is compact.

For example, all non-constant polynomial functions are proper. Exponential functions $x \mapsto e^{ax}$ are not. (But $x \mapsto \cosh(ax)$ is, if $a \neq 0$.) Clearly, bounded functions are not proper, and conversely. *Can you think of a function that's neither bounded above nor below, and yet is not proper?*

Hint: Consider $f(x) = x \sin x$.

Closedness. If $B \subset \mathbb{R}$ is closed, the preimage $f^{-1}(B)$ is in general not a closed subset of $\mathbb{R}$, and likewise the image $f(A)$ of a subset $A \subset I$ which is closed in $\mathbb{R}$ is not closed; *examples* can be given even when f is strictly monotone.

And yet if you look at the examples they are in some sense "cheating"; one is left with the feeling that, with slightly different definitions, image and preimage of closed should be closed, at least when f is monotone. And this is the definition that works:

Definition. Let $I \subset \mathbb{R}$ be open. A subset $A \subset I$ is *closed in I* if for any sequence (x_n) in I, if $\lim x_n = x$ exists and is a point in I, then also $x \in A$. (The expression "relatively closed in I" is often used, for emphasis.)

For example, the interval $A = [1, 2)$ is not closed in $\mathbb{R}$, but it is (relatively) closed in $I = [0, 2)$. (Note that "closed in $\mathbb{R}$" just means closed in the usual sense.) Note also that any subset $A \subset I$ which is closed (in the usual sense) is also closed in I.

Problem 2. Let $f : I \to \mathbb{R}$ be continuous and strictly monotone, where $I \subset \mathbb{R}$ is an open interval. Then the image $f(I) = J$ is also an open interval.

Prove:

(a) If $A \subset I$ is closed in I, then $f(A)$ is closed in J.
(b) If $B \subset J$ is closed in J, the preimage $f^{-1}(B)$ is closed in I.

Note that (a) and (b) are equivalent (via the 'lazy' argument above), so only one of them has to be proved directly.

Solution. Let's prove (b). Let (x_n) be a sequence in $f^{-1}(B)$ such that $\lim x_n = x \in I$. Then $f(x_n) \in B$ and $\lim f(x_n) = f(x) \in J$. Since B is closed in J, we have $f(x) \in B$, so $x \in f^{-1}(B)$. This shows $f^{-1}(B)$ is closed in I. (Note we do not need f to be monotone).

To prove (a) directly, take an increasing sequence (y_n) in $f(A)$ such that $\lim y_n = y \in J$. We need to show $y \in f(A)$. We have $y_n = f(x_n)$ with $x_n \in A$ and $x \in I$, $y = f(x)$. Although we don't know if $x \in A$, we do know $x_n \leq x$ for all n, since otherwise $f(x_n) > f(y)$ (since f is strictly monotone), contradicting $y_n < y$. Thus x_n is bounded above (and increasing, again by monotonicity of f), so it has a limit $\lim x_n = x_0$, with $x_0 \leq x$ and $x_0 \in I$ (since I is an interval and x_n, x are in I.) Since A is closed in I, in fact we have $x_0 \in A$. And $f(x_0) = \lim f(x_n) = \lim y_n = y$, showing that $y \in f(A)$.