HARMONIC FUNCTIONS, GREEN'S FUNCTIONS and POTENTIALS.

1. The Poisson kernel of the disk.

The *Dirichlet problem* consists of finding a harmonic function ($\Delta u = 0$) in a bounded domain in $\mathbb{R}^{\ltimes}$ with given boundary values. Physically, one seeks to find the electrostatic potential in a conductor (no charges in the interior), with its boundary held at a given potential.

Consider this problem for the disk $D_a = \{x \in \mathbb{R}^2; |x| \leq a\}$, with given boundary value h. To begin, using the Laplacian in polar coordinates (r, θ) in the plane:

$$u\Delta u(r,\theta) = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$$

we find that, for each $n \geq 1$, the functions $r^n \cos n\theta$, $r^n \sin n\theta$ are harmonic. Thus we may seek $u(r,\theta)$ of the form:

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta),$$

and substituting $u(a,\theta) = h(\theta)$ we find that A_n, B_n equal $\frac{1}{a^n}$ times the periodic Fourier coefficients of $h(\theta), \theta \in [0, 2\pi]$.

Using the definition if Fourier coefficients of h, we find:

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} [1 + 2\sum_{n=1}^{\infty} \frac{r^n}{a^n} \cos n(\theta - \phi)] h(\phi) d\phi.$$

Surprisingly, the infinite series can be summed explicitly:

$$1 + 2\sum_{n=1}^{\infty} \frac{r^n}{a^n} \cos n(\theta - \phi) = \frac{a^2 - r^2}{r^2 - 2ar\cos(\theta - \phi) + a^2}.$$

Using the Law of Cosines, we find the denominator equals the distance squared between the points $x = (r, \theta)$ (in the interior of the disk) and $y = (a, \phi)$ (on the boundary). Thus the solution may also be written (more geometrically) as an integral over the boundary, with respect to arc length $ds_y = ad\phi$:

$$u(x) = \frac{1}{2\pi a}\int_{\partial D_a} \frac{a^2 - |x|^2}{|x - y|^2} h(y) ds_y.$$

This leads to the definition of the *Poisson kernel* for the disk:

$$P(x,y) = \frac{1}{2\pi a}\frac{a^2 - |x|^2}{|x - y|^2}, \quad (|x| < a, |y| = a).$$

The Poisson kernel encodes the geometry of the domain (and the Laplacian), giving the solution of the Dirichlet problem as an integral (weighted average) of the boundary values, with 'weight function' given by $P(x,y)$:

$$u(x) = \int_{\partial D_a} P(x,y)h(y)ds_y.$$

Exercise 1. (*Poisson kernel for the exterior of the disk.*) Consider the exterior Dirichlet problem:

$$\Delta u = 0 \text{ in } \mathbb{R}^2 \setminus D_a = \{r > a\}, \quad u(x) = h(x) \text{ for } |x| = a,$$

where h is a given continuous function on the boundary of the disk. This can be solved by a similar method:
(i). Show that $r^{-n}\cos n\theta, r^{-n}\sin n\theta$ $(n \geq 1)$ are harmonic functions (for $r \neq 0$).
(ii). Show that the solution of the exterior problem is given in polar coordinates by:

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi}[1 + 2\sum_{n=1}^{\infty}\frac{a^n}{r^n}\cos n(\theta - \phi)]h(\phi)d\phi.$$

(iii) (see below.)
To sum the series, all we have to do is exchange r and a in the calculation done earlier for the interior. We obtain the expression (in geometric form):

$$u(x) = \frac{1}{2\pi a}\int_{\partial D_a}\frac{|x|^2 - a^2}{|x-y|^2}h(y)ds_y.$$

Thus the Poisson kernel for the exterior of the disk is:

$$P(x,y) = \frac{1}{2\pi a}\frac{|x|^2 - a^2}{|x-y|^2}, \quad (|x| > a, |y| = a).$$

Remark. Note that if $h \equiv 1$, this procedure would yield (in both cases) the solution $u \equiv 1$. Thus we have:

$$\frac{1}{2\pi a}\int_{\partial D_a}\frac{a^2 - |x|^2}{|x-y|^2}ds_y = 1 \text{ for } |x| < a; \quad \frac{1}{2\pi a}\int_{\partial D_a}\frac{|x|^2 - a^2}{|x-y|^2}ds_y = 1 \text{ for } |x| > a.$$

That is: in both cases the Poisson kernel defines a *probabilty distribution* on the boundary of the region, depending on the point x (an interior or exterior point.) There is a probabilistic interpretation: for any arc $A \subset \partial D_a$ of the circle, the integral of $P(x,y)ds_y$ over $y \in A$ is the probability that *Brownian motion* started at the interior point x will first hit the boundary at some point of A.

In particular we find that *this* solution of the exterior problem is bounded:

$$|u(x)| \leq ||h|| \text{ for } |x| > a, \text{ where } ||h|| = \sup_{y \in \partial D_a} |h(y)|.$$

The solution is *not unique*, however:
(iii). Show that if $u(x)$ is a solution of the exterior Dirichlet problem (with boundary value h on ∂D_r), then $v(x) = \log \frac{|x|}{a} + u(x)$ is also a harmonic function in the exterior of the disk, with the same boundary values. (Note that v is unbounded.)

Applications.

A. Mean-value property. If u is harmonic in a disk, and continuous in the closed disk, the value at the center equals the average of the boundary values. This follows directly from the fact that the value of the Poisson kernel at the origin equals $(2\pi a)^{-1}$.

If u is harmonic in the *exterior* of a disk, continuous up to the boundary and *bounded*, then it is given by integration of the boundary values with respect to the Poisson kernel, and then we have:

$$\lim_{|x| \to \infty} u(x) = \text{ average of } u \text{ over } \partial D_a.$$

This follows directly from the fact that, for the exterior Poisson kernel $P(x,y)$:

$$\lim_{|x| \to \infty} P(x,y) = \frac{1}{2\pi a},$$

uniformly over $y \in \partial D_a$, as can be seen directly from the expression above for $P(x,y)$.

Remark. The mean-value property can be used to define 'harmonic function' in a way that makes sense for continuous functions: u is harmonic if its average value over any circle equals the value at the center.

B. Strong maximum principle. Let u be harmonic in a *bounded* open region D in the plane, and continuous on the closed region $\bar{D}$. Let M be the maximum value of u over $\bar{D}$. Then u cannot take the value M at any interior point $x \in D$, unless u is constant in D.

This can be proved starting from the mean-value property (seen in class.) Note this is different from the *weak* maximum principle, which concludes only that the maximum value M must be attained at a boundary point (leaving open the possibility that it is also attained at an interior point.)

C. Differentiability. Let u be a continuous function in the closed disk $\bar{D}_a$, with the property that its value at any point is given by integration over the boundary, with respect to the Poisson kernel:

$$u(x) = \int_{\partial D_a} P(x,y)u(y)ds_y, \quad P(x,y) = \frac{1}{2\pi a}\frac{a^2 - |x|^2}{|x-y|^2}, \quad (|x| < a, |y| = a).$$

Then u is in fact smooth and harmonic ($\Delta u = 0$) in the open disk D_a. This folows from the fact that $P(x,y)$ is smooth in $x \in D_a$, for each fixed $y \in \partial D_a$. Changing coordinates to $z = x - y$, we have:

$$P(z+y,y) = \frac{1}{2\pi a}\frac{|a|^2 - |z+y|^2}{|z|^2} = -\frac{1}{2\pi a}(1 + 2\frac{z \cdot y}{|z|^2}), z \neq 0.$$

Exercise 2. Show that, for each fixed $y \in \mathbb{R}^2$, the function $w(z) = \frac{z \cdot y}{|z|^2}$ is harmonic in $\{z \in \mathbb{R}^2, z \neq 0\}$.

2. Green's indentities and applications.

Let u, v be C^2 functions in a bounded open set $D \subset \mathbb{R}^n$. Green's identities are obtained by applying the divergence theorem to the vector field $v\nabla u$, noting that:

$$div(v\nabla u) = \nabla u \cdot \nabla v + v\Delta u.$$

Denote by n the outward unit normal at points of the boundary ∂D, and let $\partial nu = \nabla u \cdot n$ be the normal derivative. The divergence theorem directly implies *Green's first indentity:*

$$\int_D v\Delta u dV = -\int_D \nabla u \cdot \nabla v dV + \oint_{\partial D} v\partial_n u dA.$$

Exchanging u and v and taking the difference, we obtain *Green's second identity*:

$$\int_D (v\Delta u - u\Delta v)dV = \oint_{\partial D}(v\partial_n u - u\partial_n v)dA.$$

(Here dV and dA denote volume integration in D, resp. integration with respect to area on ∂D.) An immediate consequence of the second identity is that *the Laplacian is a symmetric differential operator on functions of* *compact support* (i.e., vanishing outside of some large ball) with respect to the L^2 inner product: applying the identity to a large enough ball, the boundary terms vanishes and we have:

$$\langle \Delta u, v \rangle = \langle u, \Delta v \rangle.$$

Example 1. Consider the *Poisson problem* in a bounded domain $D \subset \mathbb{R}^n$, with non-homogeneous Neumann boundary conditions:

$$\Delta u = f \text{ in } D; \partial_n u = h \text{ on } \partial D.$$

This problem has physical meaning: we seek the electrostatic potential in a region with given charge density (represented by f), and given normal component of the electric field ($E_n = -\partial_n u$) on the boundary. And yet both existence and uniqueness fail! Given any solution, adding an arbitrary constant to it gives another solution (physically, the potential function is only defined up to a constant.) And Green's first identity (taking $v \equiv 1$) implies we must also have (if a solution exists):

$$\int_D f dV = \oint_{\partial D} h dA.$$

Physically this says that the total charge in D must be proportional to the average value of the normal component of the electric field on the boundary.

Mean-Value property. One may use Green's first identity to prove the mean value property for harmonic function in any dimension. Let u be harmonic in a disk $B_a = \{x \in \mathbb{R}^n; |x| < a\}$, continuous in the closed ball $\bar{B}_a$. Let $0 < R \leq a$, and consider the ball $B_R = \{r < R\} \subset B_a$. Applying the identity to this region with $v \equiv 1$, we find:

$$0 = \oint_{\partial B_R} \partial_n u dA$$

Consider what this says in terms of polar coordinates: $dA = R^{n-1} d\omega$ (where $d\omega$ is the element of area on the unit sphere $S = S^{n-1}$) and $\partial_n u$ is the radial derivative u_r:

$$\oint_S u_r(R, \omega) d\omega = 0, \quad \forall 0 < R \leq a.$$

This implies:

$$0 = \frac{d}{dr} \oint_S u(r, \omega) d\omega = \omega_{n-1} \frac{d}{dr} (\frac{1}{A(S_r)} \oint_{S_r} u dA),$$

where the last integral is the average value of u over the sphere S_r. Thus this average is constant in r, and has limit $u(0)$ as $r \to 0$. We conclude the *mean value property:*

$$u(0) = \frac{1}{A(S_a)} \oint_{S_a} u dA.$$

In words: the value of a harmonic function at a point equals its average value on any sphere centered at that point. As a consequence, the *strong maximum principle* holds for harmonic functions in all dimensions.

Remark: Note that the same argument shows that if $\Delta u \leq 0$, the average value of u over the sphere of radius r is *decreasing* in r (more precisely, non-increasing), so the value at the origin is greater than or equal to the average on S_r, for any $r > 0$. Twice-differentiable functions u satisfying $\Delta u \leq 0$ are called *superharmonic* (since they are "superaveraging"; the value at a point is greater than the average value on spheres centered at that point). Functions satisfying $\Delta u \geq 0$ are called *subharmonic*, and of course subharmonic functions are "subaveraging" in the same sense.

Uniqueness for the Poisson problem. Consider the non-homogeneous problem with Dirichlet boundary condition in a bounded domain $D \subset \mathbb{R}^n$:

$$\Delta u = f \text{ in } D, \quad u = h \text{ on } \partial D.$$

To see that there is at most one solution, consider the problem solved by the difference of two solutions:

$$\Delta w = 0 \text{ in } D, \quad w = 0 \text{ on } \partial D.$$

Then Green's first identity (setting both functions equal to w) implies:

$$0 = \int_D w \Delta w dV = -\int_D |\nabla w|^2 dV + \oint_{\partial D} w \partial_n w dA = -\int_D |\nabla w|^2 dV,$$

hence $\nabla w \equiv 0$ in D, so w is constant in D, necessarily 0.

Exercise 3. Consider the Poisson equation with Neumann boundary conditions (Example 1 above). Use Green's first identity (as above) to show that any two solutions differ by a constant.

Dirichlet's principle. (Minimizing property of harmonic functions.) It is customary to define the *energy* of a smooth function u in a bounded region $D \subset \mathbb{R}^n$ as:

$$\mathbb{E}[u] = \frac{1}{2} \int_D |\nabla u| dV.$$

(One reason for this terminology is that, in electrostatics, the energy of the electric field is proportional to $|E|^2 = |\nabla u|^2$, where u is the potential.)

It turns out that harmonic functions have the least possible energy for given boundary values, a property known as *Dirichlet's principle.* A little more precisely, if u solves the Dirichlet problem in D with boundary values h, any other function v in D with the same boundary values has energy greater than that of u.

To see this, consider $w = v - u$, which has zero boundary values. Since $v = u + w$,

$$\mathbb{E}[v] = \frac{1}{2}\int_D |\nabla u + \nabla w|^2 dV = \mathbb{E}[u] + \int_D \nabla u \cdot \nabla w dV + \mathbb{E}[w]$$

$$= \mathbb{E}[u] + \mathbb{E}[w] - \int_D (\Delta u) w dV + \oint_{\partial D} w \partial_n u dA,$$

by Green's first identity. The last two terms are zero, since u is harmonic and w has zero boundary values. Since $E[w] \geq 0$, this establishes Dirichlet's principle (and also shows u is the *unique* energy minimizer for the given boundary values.)

Exercise 4. [Strauss] There is also a "Dirichlet principle" for the Neumann problem. Let $D \subset \mathbb{R}^n$, h a given function on the boundary ∂D with zero average value ($\oint_{\partial D} h dA = 0$). Consider the "energy-type functional":

$$\mathbb{F}[u] = \frac{1}{2}\int_D |\nabla u|^2 dV - \oint_{\partial D} h u dA.$$

Let u be a harmonic function in D with $\partial_n u = h$ on ∂D. (Note that u is unique only up to a constant, but adding a constant to u doesn't change the value of $\mathbb{F}[u]$, since h has zero average.) Show that, for *any* function v on D (without restriction on boundary values or normal derivatives) we have:

$$\mathbb{F}[v] \geq \mathbb{F}[u].$$

(*Hint:* Consider $w = v - u$, and compute $\mathbb{F}[v] = \mathbb{F}[w + u]$ using Green's first identity.)

3. The Poisson problem and Green's function in $\mathbb{R}^n$.

Recall the Laplacian in polar coordinates (r, ω) is given by:

$$\Delta u = u_{rr} + \frac{n-1}{r} u_r + \frac{1}{r^2}\Delta_S u,$$

where the differential operator Δ_S acts only on the angular coordinates ω. Denote by ω_{n-1} the area of the unit sphere S^{n-1} in $\mathbb{R}^n$ (so $\omega_1 = 2\pi, \omega_2 = 4\pi$.) Define the function $G_0(x)$ in $\mathbb{R}^n$ (except for the origin):

$$G_0(x) = \frac{1}{(n-2)\omega_{n-1}r^{n-2}}(n \geq 3), \quad G_0(x) = -\frac{1}{2\pi}\log r(n=2), \quad r = |x|.$$

Exercise 5. Show that $\Delta G_0 = 0$ on $\mathbb{R}^n \setminus \{0\}$.

Now let $D \subset \mathbb{R}^n$ be a bounded domain with the origin in its interior, and (for $R > 0$ small) consider the domain $D_R = D \setminus \{r < R\}$. Applying Green's second identity to G_0 and an arbitrary function $u : D \to \mathbb{R}$ we find, since G_0 is harmonic in D_R:

$$\oint_{\partial D_R} (u\partial_n G_0 - G_0 \partial_n u)dA = -\int_{D_R} G_0 \Delta u dV.$$

Since ∂D_R is the union of ∂D and the sphere (or circle) $S_R = \partial B_R$, we may also write this as:

$$\oint_{S_R} (u\partial_r G_0 - G_0 u_r)dA - \int_{\partial D} (u\partial_n G_0 - G_0 \partial_n u)dA = \int_{D_R} G_0 \Delta u dV.$$

Our next goal is to take limits as $R \to 0$. Consider first the integral over S_R, and recall $dA = R^{n-1}d\omega$ (in polar coordinates) on S_R. Also, on S_R we have (if $n \geq 3$):

$$G_0 = \frac{1}{(n-2)\omega_{n-1}R^{n-2}}, \quad \partial_r G_0 = -\frac{1}{\omega_{n-1}R^{n-1}}.$$

Thus the integral over S_R equals:

$$-\frac{1}{(n-2)\omega_{n-1}} \oint_S u(R,\omega)d\omega + \frac{R}{(n-2)\omega_{n-1}} \oint_S u_r(R,\omega)d\omega.$$

Since $|\nabla u|$ is bounded in D, the limit of this term as $R \to 0$ is $-(n-2)u(0)$. Now consider the integral over D_R, which is the difference of the integrals of $G_0 \Delta u$ over D and over the ball B_R. This last one, written in polar coordinates, equals:

$$\frac{1}{\omega_{n-1}} \int_0^R \int_S \frac{1}{r^{n-2}} \Delta u(r,\omega) r^{n-1} d\omega dr.$$

In absolute value, this is bounded above by $\frac{R}{\omega_{n-1}} \int_0^R \int_S |\Delta u|(r,\omega)d\omega$, which clearly has limit zero as $R \to 0$. We conclude:

$$u(0) = -\int_D G_0 \Delta u dV - \oint_{\partial D} (u\partial_n G_0 - G_0 \partial_n u)dA.$$

Exercise 6. Show that the same formula is valid for $n = 2$ (using the above definition for G_0 in $\mathbb{R}^2$.)

Definition. Green's function in $\mathbb{R}^n$ with pole x_0 is defined as:

$$G_{x_0}(x) = \frac{1}{(n-2)\omega_{n-1}|x-x_0|^{n-2}}(n \geq 3), \quad G_{x_0}(x) = -\frac{1}{2\pi}\log|x-x_0|(n=2).$$

From the above, for any bounded domain D in $\mathbb{R}^n$, any C^2 function u in D and any point $x_0 \in D$, we have the **representation formula**:

$$u(x_0) = -\int_D G_{x_0}(x)\Delta u(x)dV_x - \oint_{\partial D}(u\partial_n G_{x_0} - G_{x_0}\partial_n u)dA.$$

Remark: Physically (at least for $n = 3$) G_{x_0} is the potential of a positive point charge located at x_0. (Recall the electric field equals minus the gradient of the potential, in electrostatics.)

Application: Poisson equation in $\mathbb{R}^n$. Let f be a function of compact support in $\mathbb{R}^n$ (meaning $f \equiv 0$ outside of a ball B_R of sufficiently large radius.)Suppose u is the solution of the whole-space Poisson equation:

$$\Delta u = f.$$

Assume the solution u to this problem has the property that the integrals over the sphere S_R of $u\partial_r G_x$ and of $u_r G_x$ tend to 0 as $R \to \infty$. (This is plausible on physical grounds: if the charge distribution is concentrated in a finite region of space, we expect both the potential and the electric field to decay at large distances from the charge distribution.) Applying the representation formula to $D = B_R$, and letting $R \to \infty$, this assumption implies that the boundary term vanishes in the limit, and we have, for any $x \in \mathbb{R}^n$:

$$u(x) = -\int_{\mathbb{R}^n} G_x(y)f(y)dV_y = -\int_{\mathbb{R}^n} G_x(y)\Delta u(y)dV_y.$$

This gives a *formula* for the solution of the whole-space Poisson equation (in terms of Green's function), although in a technical mathematical sense we haven't actually proved that this formula does indeed solve the problem, or that the solution does have the decay properties we assumed to derive the formula.

Remark.(Physics) Using the fact that, formally, the Laplacian is a symmetric (self-adjoint) operator, this representation formula can (informally) be written in the "equivalent" form:

$$u(x) = -\int_{\mathbb{R}^n} u(y)\Delta G_x(y)dV_y.$$

It is in this sense that "the Laplacian of Green's function with pole at x is minus the delta function at x" (if $n = 3$):

$$\Delta G_x(y) = -\delta_x(y) \quad (n = 3).$$

Maxwell's equations $div(\vec{E}) = \kappa\rho$ (where $\kappa > 0$ is a constant depending on the units, and ρ is the charge density) and $curl(\vec{E}) = 0$ (so $\vec{E} = -\nabla u$) lead to $-\Delta u = \kappa\rho$, so for Green's function with pole at x we have $\rho = \kappa\delta_x$, a positive point charge at $x \in \mathbb{R}^3$. (Though, as we saw earlier, $\Delta G_x = 0$ on $\mathbb{R}^n \setminus \{x\}$!)

Note that in some sense G_{x_0} is 'superharmonic'. First, its Laplacian is minus the delta function, so it is "negative". And it is certainly 'superaveraging', since it is infinite at the pole x_0.

Quick remark on delta functions. In the Physics literature on often sees the expression:

$$f(x) = \int_{\mathbb{R}^n} \delta(y - x)f(y)dV_y,$$

where $\delta(x - y)$ is supposed to be a "function vanishing on $\mathbb{R}^n \setminus \{x\}$" whose integral has this property for any f (in particular for $f \equiv 1$). Well, there simply is no function with this property. What we in fact have is a *linear functional* (a linear map from the space of continuous functions of compact support to the reals), denoted by δ_x:

$$\delta_x : C_c(\mathbb{R}^n) \to \mathbb{R}, \quad \delta_x[f] = f(x).$$

One reason to use the integration symbol (not a very good one) is to consider "appproximations to the delta function", for instance the heat kernel $h(x, t)$. The equality:

$$\lim_{t\to 0_+} \int_{\mathbb{R}^n} h(y - x, t)f(y)dV_y = f(x), \quad f \in C_c(\mathbb{R}^n),$$

can be interpreted as saying:

$$\lim_{t\to 0_+} h(\cdot - x, t)dV = \delta_x,$$

in the sense both sides 'have the same effect' on continuous functions f with compact support.

4. Green's function for a domain $D \subset \mathbb{R}^n$.

Definition. Let $D \subset \mathbb{R}^n$ be a bounded open set with smooth boundary, let $x_0 \in D$. *Green's function for* D with pole at x_0 is the function

$$G_{x_0}^D : D \setminus \{x_0\} \to \mathbb{R}$$

with the properties:

1. $G_{x_0}^D$ is harmonic in $D \setminus \{x_0\}$;
2. $G_{x_0}^D(y) = 0$ for $y \in \partial D$;
3. $G_{x_0}^D(y) = G_{x_0}(y) + h(y)$, where G_{x_0} is Green's function for $\mathbb{R}^n$ with pole at x_0 and h is a harmonic function in D, continuous in $\bar{D}$.

The *existence* of such a function is easy to establish, provided we can show *Dirichlet's problem* (finding a harmonic function with given boundary values) can be solved for D. All we have to do is let h be the harmonic function in D with boundary values $-G_{x_0}(y)$ on ∂D, then set $G_{x_0}^D(y) = G_{x_0}(y) + h(y)$. In the following we'll assume the Dirichlet problem can always be solved and find a formula for the solution (which can then be proven to actually yield a solution.)

Let $u : D \to \mathbb{R}$ be a smooth function, continuous on $\bar{D}$. Adding the representation formula:

$$u(x_0) = -\int_D G_{x_0}(x)\Delta u(x) dV_x - \oint_{\partial D} (u\partial_n G_{x_0} - G_{x_0}\partial_n u) dA.$$

to Green's second identity applied to the arbitrary function u and the harmonic function h:

$$0 = -\int_D h(x)\Delta u(x) dV_x - \oint_{\partial D} (h\partial_n u - h\partial_n u) dA,$$

we find (taking into account the fact that $G_{x_0}^D = G_{x_0} + h = 0$ on the boundary:

$$u(x_0) = -\int_D G_{x_0}^D(x)\Delta u(x) dV_x - \oint_{\partial D} u(\partial_n G_{x_0}^D) dA,$$

the **representation formula** for the Domain D: it 'represents' a function u in D in terms of its Laplacian and its boundary values.

Application 1. The representation formula gives an expression for the solution of the Dirichlet problem: if $\Delta u = 0$ in D and $u = h$ on ∂D, we have:

$$u(x) = \oint_{\partial D} P(x,y)h(y)dA_y, \text{ where } P(x,y) = -\partial_n G_x^D(y)$$

is the *Poisson kernel* for D.

Application 2. We also obtain a formula for the solution of the Poisson equation: $\Delta u = f$ in D, $u = h$ on ∂D:

$$u(x) = -\int_D G_x^D(y)f(y)dV_y + \oint_{\partial D} P(x,y)h(y)dA_y,$$

where $P(x,y)$ is the Poisson kernel (defined in Application 1.)

We now turn to two important examples of domains with explicitly computable Green's functions: half-spaces and balls.

Green's function for a half-space. ("Method of image charges.") Let $D \subset \mathbb{R}^n$ be the upper-half space $\{x \in \mathbb{R}^n; x_n > 0\}$, bounded by the hyperplane $H = \{x_n = 0\}$. Green's function for D with pole at $x \in D$ is (for $n \geq 3$):

$$G_x^D(y) = \frac{1}{(n-2)\omega_{n-1}}[\frac{1}{|y-x|^{n-2}} - \frac{1}{|y-\bar{x}|^{n-2}}], \quad y \in D \setminus \{x\},$$

where $\bar{x} = x - 2x_n e_n$ is the reflection of x on H.

The Poisson kernel can then be computed (note the outward unit normal to D is $-e_n$). In the following computation, we use the standard Calculus fact:

$$\frac{\partial}{\partial y_n}|y-x| = \frac{y_n - x_n}{|y-x|}, \quad \frac{\partial}{\partial y_n}|y-\bar{x}| = \frac{y_n - \bar{x}_n}{|y-\bar{x}|} = \frac{y_n + x_n}{|y-x|}, y \in H,$$

since $|y-x| = |y-\bar{x}|$ for $y \in H$.

$$P(x,y) = -\partial_n G_x^D(y) = \frac{\partial G_x^D}{\partial y_n}(y) = \frac{2}{\omega_{n-1}}\frac{x_n}{|x-y|^n}, \quad x \in D, y \in H.$$

So we have for the solution of Dirichlet's problem on the upper-half space, with boundary values h on H:

$$u(x) = \frac{2x_n}{\omega_{n-1}}\oint_H \frac{h(y)}{|x-y|^n}dA_y.$$

Remark. Note that, although we have a formula for the solution in terms of the boundary data $h(x)$, *the solution is not unique.* (This is typical for Dirichlet or Neumann problems in unbounded regions.) To see this, just note that $w(x) = x_n$ is harmonic in the upper half-space, with zero boundary values; hence can be added to any solution to produce a new solution.

On the other hand, it is true that *bounded* solutions of the Dirichlet problem are unique.

Exercise. Use the same method to compute Green's function and the Poisson kernel for the upper half-plane ($n = 2$).

Green's function for the ball. Let $D = \{x \in \mathbb{R}^n; |x| < a\}$ be the ball of radius a in $\mathbb{R}^n$. The map from $\mathbb{R}^n \setminus \{0\}$ to itself given by:

$$x \mapsto \bar{x} = \frac{a^2}{|x|^2}x, \quad (x \neq 0)$$

leaves the points on the sphere $S_a = \partial D$ fixed and is 'idempotent' (meaning $(\bar{\bar{x}}) = x$; verify!) So it is a kind of 'reflection' across the sphere (it sends $D \setminus \{0\}$ to the exterior of the sphere.)

Green's function for D is given by (for $x \in D$):

$$G_x^D(y) = \frac{1}{(n-2)\omega_{n-1}}[\frac{1}{|y-x|^{n-2}} - \frac{a^{n-2}}{|x|^{n-2}}\frac{1}{|y-\bar{x}|^{n-2}}], \quad x \neq 0, y \in D \setminus \{x\};$$

$$G_0^D(y) = \frac{1}{(n-2)\omega_{n-1}}[\frac{1}{|y|^{n-2}} - \frac{1}{a^{n-2}}], \quad y \in D \setminus \{0\}.$$

(To verify that $G_x^D(y) = 0$ if $|y| = a$, we check that $|y - \bar{x}| = a|x|^{-1}|y - x|$ if $|y| = a$.)

To compute the Poisson kernel, we find the gradient of G_x^D (using the 'Calculus fact': $\nabla_y|y - x| = \frac{y-x}{|y-x|}$):

$$\nabla_y G_x^D(y) = -\frac{1}{\omega_{n-1}|y-x|^n}(1 - \frac{|x|^2}{a^2})y, \quad x \neq 0, y \neq x;$$

$$\nabla_y G_0^D(y) = -\frac{1}{\omega_{n-1}}\frac{y}{|y|^n}, \quad y \neq 0.$$

This yields, for the Poisson kernel (since $n = y/a$ on ∂D):

$$P(x,y) = -\partial_n G_x^D(y) = -\frac{y}{a}\cdot\nabla_y G_x^D(y) = \frac{1}{\omega_{n-1}a|x-y|^n}(a^2 - |x|^2), \quad x \in D, \quad y \in \partial D.$$

Exercise. Verify this, including the fact this expression also holds for $x = 0$.

For the solution of Dirichlet's problem with boundary values h, we find, if $n \geq 3$:

$$u(x) = \frac{a^2 - |x|^2}{\omega_{n-1} a} \oint_{S_a} \frac{h(y)}{|y - x|^n} dA_y.$$

Exercise: Use the same reflection to find Green's function and the Poisson kernel for the disk of radius a, D_a (confirming the result obtained via Fourier series in section 1!)

Exercise. For any $n \geq 2$, find a harmonic function on the exterior of the ball B_a, vanishing on the boundary of the ball. Use this function to show that solutions of the exterior Dirichlet problem for the ball are not unique.