SPECTRUM OF BOUNDED DOMAINS

1. Eigenfunction expansions. Let $D \subset \mathbb{R}^n$ be a bounded open set with smooth boundary. A function u on D is an *eigenfunction* if it is not identically zero and solves the PDE on D:

$$\Delta u + \lambda u = 0$$

for some $\lambda \in \mathbb{R}$, with boundary conditions $u = 0$ (Dirichlet) or $\partial_n u = 0$ (Neumann). Eigenfunctions exist only for a discrete set of eigenvalues:

$$0 \leq \lambda_1 < \lambda_2 < \lambda_3 < \dots, \quad \lambda_n \to \infty.$$

($\lambda = 0$ occurs for Neumann boundary conditions, with *constant functions* as eigenfunctions.)

The eigenvalues can be labeled in other convenient ways. For a given eigenvalue λ, the *eigenspace*:

$$E(\lambda) = \{u \in C^2(D); \Delta u + \lambda u = 0\}$$

is a vector space of finite dimension, which may be greater than one (the dimension is the 'multiplicity' of λ.)

Eigenfunctions for different eigenvalues are orthogonal for the L^2 inner product. This follows directly from Green's second identity. If $u \in E(\lambda), v \in E(\mu)$:

$$(\mu - \lambda) \int_D uvdV = \int_D (\Delta u)v - (\Delta v)udV = \oint_{\partial D} (\partial_n u)v - (\partial_n v)udA = 0,$$

for Dirichlet or Neumann boundary conditions. So if $\lambda \neq \mu$, we have $\langle u, v \rangle = \int_D uvdV = 0$.

If $dim(E(\lambda)) > 1$, using the Gram-Schmidt process we can always find an L^2-orthogonal basis for this space. Then we may list the eigenvalues as a non-decreasing sequence:

$$0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots,$$

with a corresponding *orthogonal* set of eigenfunctions $\varphi_n \in E(\lambda_n)$ (that is, $\langle \varphi_n, \varphi_m \rangle = \int_D \varphi_n \varphi_m dV = 0$ for $m \neq n$).

The main fact of the L^2 theory is that *the set of eigenfunctions is complete* in L^2, in the sense that any function $f \in L^2(D)$ may be expanded as an infinite series of eigenfunctions:

$$f(x) = \sum_{n=1}^{\infty} A_n \varphi_n, \quad \varphi_n \in E(\lambda_n).$$

This series is convergent in the L^2 sense:

$$\int_D (f - s_N)^2 dV \to 0 \text{ as } N \to \infty, \quad s_N = \sum_{n=1}^{N} A_n \varphi_n.$$

Since the eigenfunctions φ_n are assumed to form an orthogonal set in $L^2(D)$, the coefficients A_n can be computed from:

$$\int_D f(y)\varphi_n(y) dV_y = A_n \int_D \varphi_n^2(y) dV_y = A_n ||\varphi_n||^2,$$

where $||\varphi_n||$ is the L^2 norm.

Once we have the expansion of the initial data as an eigenfunction series (which already includes the boundary conditions), it is very easy to write down formal solutions to the initial-value problems for the heat and wave equations. For the *heat equation* $u_t - \Delta u = 0$ with initial data u_0:

$$u_0 = \sum_{n \geq 1} A_n \varphi_n \Rightarrow u(x,t) = \sum_{n \geq 1} A_n e^{-\lambda_n t} \varphi_n(x),$$

convergent in $L^2(D)$, for each $t \geq 0$.

Incidentally, substituting the expression for the coefficients into the eigenfunction series, we find:

$$u(x,t) = \int_D \sum_{n \geq 1} \frac{e^{-\lambda_n t}}{||\varphi_n||^2} \varphi_n(x)\varphi_n(y) u_0(y) dV_y = \int_D h(x,y,t) u_0(y) dV_y,$$

where $h(x,y,t)$ is the *heat kernel* of D:

$$h(x,y,t) = \sum_{n \geq 1} e^{-\lambda_n t} \frac{\varphi_n(x)\varphi_n(y)}{||\varphi_n||^2}.$$

For the *wave equation* $u_{tt} - \Delta u = 0$ with initial data $(u, u_t)_{|t=0} = (u_0, u_1)$, we have:

$$u_0 = A_0 + \sum_{n \geq 1} A_n \varphi_n, \quad u_1 = B_0 + \sum_{n \geq 1} B_n \varphi_n, \quad (\lambda_n > 0 \forall n \geq 1)$$

$$\Rightarrow u(x,t) = A_0 + B_0 t + \sum_{n\geq 1} A_n \cos(\sqrt{\lambda_n} t)\varphi_n(x) + \frac{B_n}{\sqrt{\lambda_n}} \sin(\sqrt{\lambda_n} t)\varphi_n(x).$$

2. Spectrum of the disk. Let $D_a = \{x \in \mathbb{R}^2; |x| < a\}$ be the disk of radius a. We compute the Dirichlet spectrum:

$$\Delta u + \lambda u = 0 \text{ in } D_a, \quad u = 0 \text{ on } \partial D_a.$$

The Laplacian in polar coordinates (r, θ) is:

$$\Delta u = u_r r + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}.$$

Since we know the eigenvalues of the operator $L[u] = u_{\theta\theta}$, it is natural to look for solutions of the form:

$$u(r,\theta) = f(r) Y_n(\theta), \quad Y_n(\theta) = A\cos n\theta + B \sin n\theta (n \geq 0) \text{ so } (Y_n)_{\theta\theta} = -n^2 Y.$$

We obtain for $f(r)$ the ordinary differential equation:

$$f_{rr} + \frac{1}{r} f_r + (\lambda - \frac{n^2}{r^2}) f = 0, \quad r > 0.$$

This O.D.E. depends on two parameters (λ and n), but it turns out the parameter λ may be 'scaled away'. The change of variable $x = \sqrt{\lambda} r$ leads to the O.D.E. in the new variable:

$$f_{xx} + \frac{1}{x} f_x + (1 - \frac{n^2}{x^2}) f = 0, \quad x > 0.$$

This classical O.D.E is known as 'Bessel's equation with parameter n. Like any self-respecting second-order linear O.D.E. it admits two linearly independent solutions. Only one is finite at $x = 0$, the *Bessel function of order* n, $J_n(x)$, given by a convergent power series at zero of the form:

$$J_n(x) = c_n x^n [1 + \sum_{j=1}^{\infty} a_j^{(n)} x^{2j}], \quad n \geq 0.$$

The normalization constant c_n equals $(2^n n!)^{-1}$, but that's not important here.

Exercise: The fact that the eigenvalue can be 'scaled away' is easily seen in general: if a function $u(x)$ in $\mathbb{R}^n$ is a solution of $\Delta_x u + \lambda u = 0$, and we rescale the independent variable by setting $y = \sqrt{\lambda}x$, show that the function $v(y) = u(\frac{y}{\sqrt{\lambda}})$ satisfies $\Delta_y v + v = 0$. (Here Δ_x, Δ_y denote the Laplacians in the variables x, y (resp.)

The Bessel functions $J_n(x)$ are oscillatory with decreasing amplitude, as seen from the asymptotics:

$$J_n(x) \sim (\frac{2}{\pi})^{-1/2}\frac{1}{\sqrt{x}}\cos(x - \frac{\pi}{4} - \frac{n\pi}{2}), \quad \text{as } x \to \infty.$$

In particular $J_n(x)$ has infinitely many positive (simple) zeros:

$$0 < x_{n1} < x_{n2} < \dots, \quad J_n(x_{nk}) = 0.$$

From the Dirichlet boundary condition we obtain $J_n(\sqrt{\lambda}a) = 0$. This means we should label the eigenvalues by two indices n, k: n labels the parameter of the Bessel function and k labels the zeros of J_n.

To summarize, we consider two cases:

(1) *Rotationally symmetric case.* Only J_0 is involved. The eigenvalues/eigenfunctions are:

$$\lambda_{0k} = \frac{x_{0k}^2}{a^2}, \quad u_{0k}(r) = J_0(\frac{x_{0k}r}{a}), \quad k = 1, 2, \dots.$$

The eigenspaces are one-dimensional in this case. A general radial function in D_a admits an expansion:

$$f(r) = \sum_{k=1}^{\infty} A_k u_k(r), \quad \int_0^a f(r)u_k(r)rdr = A_k \int_0^a u_k^2(r)rdr.$$

(2) *General case.* The eigenvalues/eigenfunctions are labeled by two indices:

$$\lambda_{nk} = \frac{x_{nk}^2}{a^2}, \quad u_{nk}(r,\theta) = J_n(\frac{x_{nk}r}{a})(A\cos n\theta + B\sin n\theta),$$

for $n \geq 0$ and $k \geq 1$. The eigenspaces $E(\lambda_{0k})$ are one-dimensional, but the eigenvalues λ_{nk} have multiplicity two for $n \geq 1$. A general function $f(r,\theta)$ admits an expansion:

$$f(r,\theta) = \sum_{k=1}^{\infty} C_k u_{0k}(r) + \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} J_n(\frac{x_{nk}r}{a})(A_{nk}\cos n\theta + B_{nk}\sin n\theta),$$

where the coefficients can be obtained using orthogonality of eigenfunctions.

3. Spectrum of the ball. Turning to $D = B_a = \{x \in \mathbb{R}^3; |x| < a\}$, we proceed in the same fashion. The Laplacian in polar coordinates (r, ω) ($r > 0$, $\omega \in S$, the unit sphere) is:

$$\Delta u = u_{rr} + \frac{2}{r} u_r + \frac{1}{r^2} \Delta_S u,$$

where $\Delta_S u$ is the 'spherical Laplacian': a second-order differential operator acting only on the 'angular coordinates' ω. This operator has its own eigenvalues γ and eigenfunctions $Y(\omega)$, functions on S satisfying:

$$\Delta_S Y + \gamma Y = 0 \text{ on } S. \quad \text{Notation: } Y \in E_S(\gamma).$$

No boundary conditions are needed, since S is a closed surface without boundary. (We'll say more about the spectrum of Δ_S shortly.) As before, we look for eigenfunctions of Δ of the form (in polar coordinates):

$$u(r, \omega) = f(r) Y(\omega), \quad Y \in E_S(\gamma),$$

and then $\Delta u + \lambda u = 0$ leads to the ordinary differential equation for f:

$$f_{rr} + \frac{2}{r} f_r + (\lambda - \frac{\gamma}{r^2}) f = 0.$$

This *almost* looks like Bessel's equation. To turn the coefficient $\frac{2}{r}$ to $\frac{1}{r}$ we make a change of (dependent) variable:

$$w(r) = \sqrt{r} f(r) \Rightarrow w_{rr} + \frac{1}{r} w_r + (\lambda - \frac{\gamma + \frac{1}{4}}{r^2}) w = 0.$$

Scaling away the eigenvalue as before, we find $w(r) = J_s(\sqrt{\lambda} r)$, $s = \sqrt{\gamma + \frac{1}{4}}$, where $J_s(x)$ is a solution (finite at $x = 0$) of *Bessel's equation with parameter* $s \in \mathbb{R}_+$:

$$g_{xx} + \frac{1}{x} g_x + (1 - \frac{s^2}{x^2}) g = 0, \quad x > 0.$$

The power series expansion and asymptotics are identical to the case of integer parameter:

$$J_s(x) = c_s x^s [1 + \sum_{j=1}^{\infty} a_j^{(s)} x^{2j}], \quad x > 0, c_s = \frac{1}{2^s \Gamma(s+1)};$$

$$J_s(x) \sim (\frac{2}{\pi})^{1/2} \frac{1}{\sqrt{x}} \cos(x - \frac{s\pi}{2} - \frac{\pi}{4}) \text{ as } x \to \infty.$$

The Bessel functions have the useful *parameter shifting* property:

$$J_{s+1}(x) = \frac{s}{x} J_s(x) - J_s'(x).$$

Since the values of s for this problem depend on the eigenvalues γ of the spherical Laplacian Δ_S, we describe it spectrum.

Spectrum of the spherical Laplacian.

The eigenvalues are $\{\gamma_l = l(l+1), l = 0, 1, 2, \ldots\}$.

The eigenspace $E(\gamma_l)$ has dimension $2l + 1$.

A basis of eigenfunctions is labeled $Y_{lm}(\omega), m = -l, \ldots, 0, \ldots, l$.

(We'll see later that $E(\gamma_l)$ has a remarkable alternative description.) For now, notice that this gives for the Bessel equation parameter s:

$$s = \sqrt{l(l+1) + \frac{1}{4}} = l + \frac{1}{2}, l = 0, 1, 2, \ldots$$

We now put it all together to describe the spectrum of the Laplacian on B_a, with Dirichlet boundary conditions. For each fixed natural number $l \geq 0$, consider the sequence of positive zeros of $J_{s+\frac{1}{2}}$:

$$0 < \rho_{l1} < \rho_{l2} < \rho_{l3} < \ldots, \quad J_{l+\frac{1}{2}}(\rho_{lk}) = 0, k = 1, 2, \ldots.$$

Then the Dirichlet eigenvalues are determined by the condition $J_{l+\frac{1}{2}}(\sqrt{\lambda} a) = 0$, and hence depend on two indices: l and k:

$$\lambda_{lk} = \frac{\rho_{lk}^2}{a^2}.$$

The eigenspaces $E(\lambda_{0k})$, corresponding to rotationally symmetric eigenfunctions, have multiplicity 1, with a basis consisting of

$$u_k(r) = J_{\frac{1}{2}}(\frac{\rho_{0k} r}{a}).$$

On the other hand, for $l > 0$ the eigenspaces $E(\lambda_{lk})$ have dimension $2l + 1$, with a basis given by:

$$u_{l,k,m}(r, \omega) = J_{l+\frac{1}{2}}(\frac{\rho_{lk} r}{a}) Y_{lm}(\omega), \quad m = -l, \ldots, l.$$

The corresponding eigenfunction expansions are easily written down. In the rotationally symmetric case:

$$f(r) = \sum_{k=1}^{\infty} A_k J_{\frac{1}{2}}(\frac{\rho_{0k} r}{a}),$$

where, by orthogonality:

$$\int_0^a f(r) J_{\frac{1}{2}}(\frac{\rho_{0k} r}{a}) r^2 dr = A_k \int_0^a [J_{\frac{1}{2}}(\frac{\rho_{0k} r}{a})]^2 r^2 dr = A_k a^3 \int_0^1 J_{\frac{1}{2}}(\rho_{0k} s) s^2 ds.$$

In the general case:

$$f(r,\omega) = \sum_{k=1}^{\infty} C_k J_{\frac{1}{2}}(\frac{\rho_{0k} r}{a}) + \sum_{l=1}^{\infty} \sum_{k=1}^{\infty} \sum_{m=-l}^{l} A_{lkm} J_{l+\frac{1}{2}}(\frac{\rho_{lk} r}{a}) Y_{lm}(\omega),$$

with the coefficients $C_k, A_{l,k,m}$ computable by orthogonality.

So far, so good. But there are two surprises in store. The first one is discovered by considering a change of variable that eliminates the first-derivative term in Bessel's equation with parameter s. (Such a change of variable exists for any second order O.D.E.) Setting $u(x) = \frac{1}{\sqrt{x}} v(x)$, we find:

$$u_{xx} + \frac{1}{x} u_x + (1 - \frac{s^2}{x^2}) u = 0 \Rightarrow v_{xx} + (1 + \frac{s^2 - \frac{1}{4}}{x}) v = 0.$$

Exercise: verify this.

But this means something special happens when $s = 1/2$! We obtain the elementary O.D.E. $v_{xx} + v = 0$, with general solution $v(x) = A\cos x + B\sin x$. Since we want $u(x)$ to be finite at zero the $\cos x$ component is out; using the conventional normalization constant $\sqrt{2/\pi}$:

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x,$$

with positive zeros: $\rho_{0k} = k\pi$.

And then we realize that the recursion formula for $J_{s+1}(x)$ in terms of $J_s(x)$ seen earlier implies *all the Bessel functions of half-integral order are elementary functions!* For instance:

$$J_{\frac{3}{2}}(x) = \frac{1}{2x} J_{\frac{1}{2}}(x) - J'_{\frac{1}{2}}(x)$$

$$= \sqrt{\frac{2}{\pi}}[\frac{\sin x}{x^{3/2}} - \frac{\cos x}{x^{1/2}}].$$

Exercise. (i) Verify this expression for $J_{\frac{3}{2}}$. (ii) Use it to obtain the asymptotics for $x \to \infty$ described above. (iii) Verify that, for $x \sim 0$:

$$J_{\frac{1}{2}}(x) \sim (const.)x^{1/2} \text{ and } J_{\frac{3}{2}}(x) \sim (const.)x^{3/2}.$$

The last surprising fact in store is that there is a beautiful algebraic description of the eigenfunctions $Y_{lm}(\omega)$ of the 'spherical Laplacian' operator Δ_s: it turns out the functions $r^l Y_{lm}(\omega)$, when expressed in rectangular coordinates x, y, z are *homogeneous harmonic polynomials of degree l!* Part of this is easy to verify:

Exercise: Use the expression for the Laplacian in polar coordinates (r, ω) in $\mathbb{R}^3$ to show that:

$$\Delta_S Y(\omega) = -l(l+1)Y(\omega) \text{ implies } \Delta(r^l Y(\omega)) = 0,$$

and conversely if $r^l Y(\omega)$ is harmonic, then $\Delta_S Y + l(l+1)Y = 0$.

The difficult part is showing that, when expressed in rectangular coordinates, the functions $r^l Y(\omega)$ are polynomials of pure degree l in x, y, z. Stated in linear algebra terms, we have two vector spaces: the space $\mathcal{P}_l$ of homogeneous polynomials of degree l in x, y, z; and the subspace $\mathcal{H}_l \subset \mathcal{P}_l$ of homogeneous *harmonic* polynomials of degree l. The Laplacian defines a linear map:

$$\Delta : \mathcal{P}_{l+2} \to \mathcal{P}_l,$$

and the kernel (or nullspace) of this linear map is exactly $\mathcal{H}_{l+2}$. Clearly $\mathcal{P}_0 = \mathcal{H}_0$ consists of the constants, while $\mathcal{P}_1 = \mathcal{H}_1$ is spanned by $\{x, y, z\}$.

$$\mathcal{P}_2 = span\{x^2, y^2, z^2, xy, xz, yz\}$$

is six-dimensional. We expect $\mathcal{H}_2$ to be five-dimensional (since $l = 2, 2l+1 = 5$.) It is easy to check that the homogeneous polynomials of degree 2:

$$\{xz, xy, yz, x^2 - y^2, 2z^2 - x^2 - y^2\}$$

are harmonic and linearly independent, and hence form a basis for $\mathcal{H}_2$. In general we have in $\mathbb{R}^3$:

$$dim\mathcal{P}_l = \frac{(l+1)(l+2)}{2}, \quad dim\mathcal{H}_l = 2l + 1.$$

Remark 1: Dimensional count. The dimension of $\mathcal{P}_l$ is found by a purely combinatorial argument: it is the number of ways to place 2 'sticks' into $l+2$ 'gaps'. On the other hand, it can be shown that we have the following decomposition of $\mathcal{P}_l$ into complementary subspaces:

$$\mathcal{P}_l = \mathcal{H}_l \oplus (x^2+y^2+z^2)\mathcal{P}_{l-2}, \quad l \geq 2.$$

And hence, for the dimensions we have:

$$dim\mathcal{H}_l = dim\mathcal{P}_l - dim\mathcal{P}_{l-2} = \frac{1}{2}[(l+2)(l+1) - l(l-1)] = 2l+1.$$

Remark 2: Conventional labeling. The choice of index m in labeling a basis $\{r^l Y_{lm}\}$ of $\mathcal{H}_l$ comes from Quantum Mechanics, and it is related to the degree of z in the polynomial: $|m|$ equals l minus the degree of z. So when $l=2$:

$$E_2 = span\{x^2-y^2, xy\}(m=\pm 2), E_1 = span\{xz, yz\}(m=\pm 1), E_0 = span\{2z^2-x^2-y^2\}(m=0).$$

(Alternatively, $l-|m|$ is the eigenvalue of the operator $z\partial_z$ acting on $\mathcal{H}_l$.)

Remark 3: It is easy to describe the space $\mathcal{H}_l$ of harmonic polynomials of degree l in x, y for $\mathbb{R}^2$: it is two-dimensional, spanned by the real and imaginary parts of z^l, where $z = x+iy$. The reason is that polar coordinates correspond to the modulus-phase Gauss form:

$$z = x+iy = re^{i\theta} \Rightarrow z^l = r^l e^{il\theta} = r^l \cos\theta + ir^l \sin l\theta,$$

where the real and imaginary parts $r^l \cos l\theta, r^l \sin l\theta$ are harmonic functions. For example:

$$\mathcal{H}_3 = span\{x^3 - 3xy^2, 3x^2y - y^3\} = span\{Re(z^3), Im(z^3)\}.$$

Remark 4: All of this generalizes to higher dimensions.

Problems.

1. Write down the solution of the wave equation $u_{tt} - \Delta u = 0$ on the unit disk $D_1 \subset \mathbb{R}^2$, with Dirichlet boundary conditions $u = 0$ at $r = 1$, and initial data:

$$u(x, 0) = 1 - |x|^2, \quad u_t(x, 0) \equiv 0, x \in D.$$

2. Find the solution of the heat equation $u_t - \Delta u = 0$ on the unit disk, with *Neumann* boundary conditions $u_r = 0$ at $r = 1$, and initial condition $u_0(r)$ (arbitrary), $r = |x|$.

3. Find the solution of the diffusion equation $u_t - \Delta u = 0$ in the disk $\{r < a\}$ in $\mathbb{R}^2$, with Dirichlet boundary conditions ($u = 0$ at $r = a$) and initial data $u_0(r)$ (arbitrary).

4. Find the *Neumann* eigenvalues/eigenfunctions for the Laplacian on the unit ball B_1 in $\mathbb{R}^3$, acting on *radial* functions $u(r)$. *Hint:* Let $v(r) = ru(r)$, and find the equation satisfied by $v(r)$.

5. Solve the diffusion equation $u_t - \Delta u = 0$ on the unit ball in $\mathbb{R}^3$, with Neumann boundary conditions $u_r = 0$ and arbitrary radial initial condition $u_0(r)$. (See problem 4 for a hint.)

6. Solve the diffusion equation $u_t - \Delta u = 0$ on a ball of radius 1 in $\mathbb{R}^3$, with $u \equiv A$ on the boundary and $u \equiv U_0$ (a constant) when $t = 0$. (The solution is radial.)

7. Solve the wave equation $u_{tt} - \Delta u = 0$ on the unit ball in $\mathbb{R}^3$, with Neumann boundary conditions $u_r = 0$ and initial data $u(x, y, z, 0) = z, u_t \equiv 0$.

8. Find the *bounded* harmonic function in the *exterior* $\{r > 1\}$ of the unit ball in $\mathbb{R}^3$, with the boundary condition $u_r(x, y, z) = z$ on $r = 1$.